An Obstacle Problem for Mean Curvature Flow

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Basic Notation

List of frequently used sets.

Symbol:	Definition:	Name / Remarks:
$\mathbb{N}$	$\{1,2,3,\dots\}$	natural numbers
$\mathbb{N}_0$	$\{0,1,2,3,\dots\}$	natural numbers including zero
$]a,b[$	$\{x\in\mathbb{R} : a<x<b\}$	open interval; $a,b\in\mathbb{R}$
$[a,b]$	$\{x\in\mathbb{R} : a\le x\le b\}$	closed interval; $a,b\in\mathbb{R}$
$[x;y]$	$\{x+t(y-x)\in\mathbb{R}^n : t\in[0,1]\}$	segment; $x,y\in\mathbb{R}^n$
$B_r(x;\mathbb{R}^n)$	$\{y\in\mathbb{R}^n \,:\, \lvert x-y\rvert<r\}$	$x\in\mathbb{R}^n$, $r>0$
Ω_r	$\{x\in\Omega \,:\, \inf_{y\in\partial\Omega}\lvert x-y\rvert>r\}$	$\Omega\subset\mathbb{R}^n$, $t>0$
$\partial_p(\Omega\times[a,b[)$	$(\Omega\times\{a\})\cup(\partial\Omega\times[a,b])$	parabolic boundary

Lipschitz functions. We say a function $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ is Lipschitz continuous if there exists a constant $C\ge 0$ such that $|f(x)-f(y)|\le C|x-y|$ for every $x,y\in\Omega$ and the smallest such constant – the Lipschitz constant of f – shall be denoted by $\mathrm{Lip}(f)$. The space of all Lipschitz functions (defined on Ω) will be denoted by $\mathrm{Lip}(\Omega)$. Since every Lipschitz function $f:\Omega\to\mathbb{R}$ can be extended uniquely to $\overline{\Omega}$, by a slight abuse of notation, we will often use the expression f for this extension as well and, moreover, we will use $f(x)$ to denote the value of this extension at points $x\in\partial\Omega$. In other words, we implicitly assume that Lipschitz functions are defined up to the boundary.

Hölder Continuity. For $\alpha\in[0,1]$ we say that $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ is (α-)Hölder continuous, if there exists some constant $C>0$ such that $|u(x)-u(y)|\le C|x-y|^\alpha$ for every $x,y\in\Omega$ and for bounded Ω, the space of all those functions shall be denoted by $C^{0,\alpha}(\Omega)$ throughout this thesis. Note, that in the literature, these spaces are sometimes denoted by $C^{0,\alpha}(\overline{\Omega})$. To make things more confusing, the local versions of these spaces, that we shall call $C^{0,\alpha}_{loc}(\Omega):=\{u:\Omega\to\mathbb{R}\,|\,u\in C^{0,\alpha}(V)\ \forall V \text{ open, with } \overline{V}\subset\Omega\}$, might be denoted by other authors by $C^{0,\alpha}(\Omega)$.

Mollifiers. Often, we will make use of the so called technique of *mollification*, for further details see for instance [45, Section 4.2.1]. Here, we only want to note, that for $\varepsilon>0$, by the so called *standard mollifier* $\rho_\varepsilon:\mathbb{R}^n\to\mathbb{R}$ we mean the function given by $\rho_\varepsilon(x):=\frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$ where for $x\in\mathbb{R}^n$

$$\rho(x):=\begin{cases} C(n)\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1,\\ 0 & \text{else.}\end{cases}$$

The normalization constant $C(n)$ is chosen such that $\|\rho\|_{L^1(\mathbb{R}^n)}=1$.

Finally, we also refer to [45] for any ambiguity concerning the notation and properties of Sobolev spaces.

Introduction

In order to introduce the reader to the subject of this thesis, we would like to explain and highlight some of the features of obstacle problems on the probably most prominent example. Namely, the so called *classical obstacle problem*, which in its simplest formulation consists of finding a function $u : \Omega \to \mathbb{R}$ minimizing the Dirichlet integral

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x,$$

in some domain $\Omega \subset \mathbb{R}^n$, among all functions $u \in W_0^{1,2}(\Omega)$ such that $u \geq \psi$ in Ω for some given $\psi : \Omega \to \mathbb{R}$. The function ψ is called the obstacle and if one thinks of the graph of u representing a membrane, this condition tells that the membrane is not allowed to penetrate the solid obstacle described by ψ. In contrast to the unconstrained case, the (unique) solution to this problem will no longer be smooth. In fact, solutions can have discontinuous second order derivatives even if one assumes that $\partial\Omega$ and ψ are smooth. An equivalent way to formulate this particular problem would be: Find $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ such that

$$u \geq \psi, \quad -\Delta u \geq 0 \quad \text{and} \quad \Delta u = \chi_{\{u=\psi\}} \Delta\psi = \begin{cases} 0 & \text{in } u > \psi, \\ \Delta\psi & \text{in } u = \psi. \end{cases}$$

This formulation is closer to the usual formulation of partial differential equations but actually retains some peculiarities. Indeed, one of the main differences is the presence of the so called coincidence set $\{u = \psi\}$ which is a part of the unknowns in obstacle problems. The set $\partial\{u = \psi\} \cap \Omega$ is called the *free boundary* and its analysis is an important question on its own. Finally, a third possible way to formulate this problem is: Find $u \in W_0^{1,2}(\Omega)$ with

$$\int_\Omega \nabla u \cdot \nabla(v - u) \, \mathrm{d}x \geq 0, \quad \forall v \in W_0^{1,2}(\Omega) \text{ with } v \geq \psi.$$

Such inequalities are called *variational inequalities* and they are the counterparts of the Euler-Lagrange equations for usual partial differential equations. For much more on this classical problem, we refer to [25].

Obstacle problems, or more generally *free boundary problems* arise in many fields of pure and applied mathematics. Here, we only mention potential theory, control theory, mathematical biology and finance as examples and refer for instance to the monograph [113] for a thorough introduction to obstacle problems in mathematical physics or the classic references [55] and [87] for an introduction to free boundary problems.

From a geometric point of view, the most natural nonlinear obstacle problem is arguably the one for minimal surfaces. This means that instead of minimizing the Dirichlet energy, one now tries to find minima of the energy

$$\int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x,$$

i.e. one is looking for the function u whose graph has the smallest surface area among all competitors who lie above the obstacle and satisfy some given boundary conditions. Indeed, this problem has been studied extensively in the past and to mention some important contributions we cite the works of Nitsche [110], Giaquinta-Pepe [66], Lewy-Stampacchia [91] Miranda [106], Giusti [69, 70], Kinderlehrer [85], and Friedrich [121].

Of course, one can also consider obstacle problems for parabolic equations. For the simplest case of the heat equation we refer for instance to Brezis [19] or Friedman [53], [55, Section 1.8]. Having mentioned parabolic variational inequalities, we also recall the connection with the famous Stefan problem. Roughly speaking, it is a mathematical model for melting ice which is in contact with water. Indeed, Duvaut [40] transformed the one-phase Stefan problem into a variational inequality. Being the archetype of a free boundary problem, also the Stefan problem was studied extensively in the literature. We can again just give an incomplete list of references and mention Rubinstein's monograph [115], Friedman [51, 52], Friedman-Kinderlehrer [56], and Luckhaus [95]. For some historical remarks on the Stefan problem, we refer to [124].

Also due to its applications in the pricing of American options in mathematical finance parabolic obstacle problems draw a lot of interest, see for instance [13, 14, 15, 17, 81, 111]. More recently, in the same context, also the study of degenerate parabolic obstacle problems was intensified see for instance Daskalopoulos-Feehan [37, 38], or, in another context, in the elliptic case Danielli-Garofalo-Salsa [36].

In this thesis we are going to study a parabolic, non-linear, degenerate analog of the classical obstacle problem. Namely, we will investigate the obstacle problem for the mean curvature operator. More precisely, we study the evolution of a surface, initially given as the graph of some smooth function $u_0 : \Omega \to \mathbb{R}$ (defined on some bounded, smooth and convex domain $\Omega \subset \mathbb{R}^n$) which is constrained to stay above some given, smooth $\psi : \Omega \to \mathbb{R}$ for all times. Roughly speaking, we look for some $u : \Omega \times [0, +\infty[\to \mathbb{R}$ with $u(\cdot, 0) = u_0$, $u(x,t) \geq \psi(x)$ and satisfying the equation

$$u_t - \sqrt{1+|\nabla u|^2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \tag{0.1}$$

at every point $(x,t) \in \Omega\times]0,+\infty[$ with $u(x,t) > \psi(x)$ while at points (x,t) of the coincidence set Λ given by

$$\Lambda := \{(x,t) \in \Omega \times [0,+\infty[:\, u(x,t) = \psi(x)\},$$

equation (0.1) holds only if this leads to a detachment from the graph of ψ, otherwise the surface remains still at this point. Additionally, we impose the boundary condition

$u(\cdot,t)_{|\partial\Omega} = u_{0|\partial\Omega}$ for all $t > 0$. Even more heuristically, one can think of a thin, elastic membrane which is attached to a fixed, closed wire along its boundary and trying to attain a configuration of minimal area. However, this membrane is constrained to stay above a fixed, solid *obstacle* described by ψ. Therefore, the points on the surface, that do not touch the obstacle move with velocity equal to the mean curvature vector at this point. As soon as the surface touches the obstacle, these points remain still until the mean curvature vector points away from the obstacle at some later time, in which case the surface is allowed to detach from the obstacle again. Putting these heuristic remarks in one formula, we would get

$$u_t(x,t) = \begin{cases} \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) & \text{if } u(x,t) > \psi(x), \\ \max\left\{0, \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right\} & \text{if } u(x,t) = \psi(x). \end{cases}$$

This problem, on the one hand, can be considered as an extension of the classical non-parametric mean curvature flow with Dirichlet boundary conditions that was pioneered by Lieberman [92] and Huisken [77]. On the other hand, this problem is also a parabolic analog of hypersurfaces of prescribed mean curvature over obstacles that were studied for instance by Gerhardt [61] and Mazzone [101] (and in the special case of mean curvature equal to zero in the literature on constrained minimal surfaces mentioned above). As in the case of the classical obstacle problem, the presence of an obstacle will lead to an evolution which in general will not be smooth and hence any (smooth) parametric approach will break down when the surface touches the obstacle.

Typically, and as shown above for the classical obstacle problem, a precise way to formulate an obstacle problem is to write it in form of a so called *variational inequality*. For the present problem, we will derive in Theorem 3.4.8 the variational inequality

$$\int_0^{+\infty}\int_\Omega u_t\phi\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^{+\infty}\int_\Omega \sqrt{1+|\nabla u|^2}\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\phi\,\mathrm{d}x\,\mathrm{d}t, \tag{0.2}$$

which holds for every $\phi \in C^1_c(\Omega \times [0,+\infty[)$ with $\phi \geq 0$ on the coincidence set Λ. As we can see, due to the non-divergence form of (0.1) a formulation as given in (0.2) does not allow to pass one spatial derivative to the test-function and thus requires u to have derivatives up to second order in space (at least in a weak sense). Moreover, the operator we consider is not linear but only quasi-linear and degenerates as $|\nabla u| \to +\infty$.

The approach that we choose is via a time discretization scheme that was introduced for mean curvature (and related) flows by Almgren-Taylor-Wang [3] and Luckhaus-Sturzenhecker [96]. Initially, the scheme was suggested as a possible weak definition for mean curvature flow and was a main source of inspiration for the definition of generalized minimizing movements by De Giorgi [39]. Apart from earlier works in material science, mean curvature flow was first studied by Brakke [18] in his PhD-Thesis with the aim of modeling the motion of grain boundaries (See also the recent simplification of

Brakke's approach by Kasai-Tonegawa [83]). Brakke was using the notion of varifolds to describe the moving surfaces which allowed him to also deal with emerging singularities that one would expect from observation of actual grain boundaries. A few years later, the theory of (parametric) mean curvature flow in a smooth setting was initiated and developed with enormous success. Among many other important papers we mention the ones by Gage-Hamilton [57], Grayson [73], Huisken [76, 78], Huisken-Ecker [42, 43] and refer to the survey paper [34] for a much more detailed list of references, also including recent developments on smooth mean curvature flow. Despite the success of this approach, in many situations where, as in Brakke's original work, the problem under consideration arises from a physical model it would be desirable to find a description which will not stop as soon as singularities occur but continue to describe the motion. For this reason, many attempts at the study of mean curvature flow in a non-smooth setting have been made and apart from the already mentioned time discretization method and Brakke's approach we would like to mention the approach by Evans-Spruck [46, 47, 48, 49] and Chen-Giga-Goto [33], using methods from the theory viscosity solutions and a level-set formulation and Illmanen's elliptic regularization [79, 80]. Of course, this list is far from complete and the interested reader is referred to the literature on mean curvature flow, contained in the monographs [10], [41] or [98].

The primary focus of this thesis is to investigate the time discretization scheme further and adapt it properly in order to prove existence of solutions to the previously described parabolic obstacle problem and derive and discuss further properties of those solutions. To this regard, we stress the similarities with the Luckhaus-Sturzenhecker approach which inspired our work. Indeed, the time discretization proposed by Almgren-Taylor-Wang and Luckhaus-Sturzenhecker in principal does not satisfy any equation (unless a classical solution is known to exist). It was one of the main achievements of [96] to deduce that the solutions produced by the approximation scheme actually solve an equation in a distributional sense. However, their derivation was made under an additional assumption which excludes a loss of energy in the limit. We recover the distributional formulation of Luckhaus and Sturzenhecker in the form of the variational inequality (0.2). Furthermore, due to the specific geometric assumptions that we made (graphical initial surface), we are able to derive unconditional convergence of the scheme, i.e. without any further assumptions. Of course, we are not the first to apply this time discretization to other problems. Quite the contrary, it is impossible to give an exhaustive list of all the work that was done using (adapted) versions of this schemes. We only mention the work by Chambolle and co-authors [2, 11, 12, 27, 28, 29, 30, 31, 32], the work by Röger [114] and Abels-Röger [1], Balzani-Rumpf's work on Willmore flow [7], the recent preprints by Spadaro [120] and Mugnai-Seis-Spadaro [109] and finally the recent work by Esedoḡlu-Otto [44] and Laux-Otto [90], on questions related to a thresholding scheme introduced by Merriman-Bence-Osher [104, 105], see also the recent preprints by Tonegawa-Wickramasekera [122] and Kim-Tonegawa [84].

A detailed introduction to the time discretization scheme will be contained in chapter one. Here, we would just like to illustrate in which sense this approach overcomes the difficulties mentioned above. Loosely speaking, the scheme transforms our parabolic problem into a sequence of elliptic problems into which one can easily incorporate the obstacle and the boundary conditions. More precisely, if, for a given time step size $h > 0$, the (approximate) solution at time $t = kh$ (for some $k \in \mathbb{N}$) is given by $u_k : \Omega \to \mathbb{R}$, to obtain u_{k+1} (i.e. the approximate solution at time $t = (k+1)h$) we will have to solve an obstacle problem of prescribed mean curvature type. This means, that we are solving problems of the type: Find $u : \Omega \to \mathbb{R}$, minimizing an energy of the form

$$\int_\Omega \sqrt{1 + |\nabla u|^2} \,\mathrm{d}x + \int_\Omega \int_0^z H(x,z) \,\mathrm{d}z \,\mathrm{d}x,$$

among all u which lie above the obstacle and are subject to Dirichlet boundary conditions, with $H : \Omega \times \mathbb{R} \to \mathbb{R}$ some Lipschitz continuous function depending on u_k. The associated Euler-Lagrange equation of this problem is (in the unconstrained case)

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = H(x, u(x)).$$

We note, that this is still a non-linear, (non-uniformly) elliptic equation with a non-linear right hand side. However, this problem is now of divergence form which allows to apply classical techniques to find solutions. For a given time step size $h > 0$ we will thus get a sequence of functions $(u_k^h)_{k \in \mathbb{N}}$ defined on Ω which all respect the obstacle and give rise (via constant-in-time interpolation) to a so called approximate solution $u^h : \Omega \times [0, +\infty[\to \mathbb{R}$. The main difficulty lies then first of all in the task of finding estimates which allow to pass into the limit $h \to 0$ and secondly to investigate the properties of such limits. Eventually, we will be able to show that this limit is unique and solves the obstacle problem in the sense of the variational inequality (0.2).

Related Works on Mean Curvature Flow with Obstacles

Recently, mean curvature flow with obstacle aroused quite some interest. Let us therefore also mention some ongoing research on related problems. Almeida, Chambolle and Novaga study (parametric) mean curvature flow with obstacles in [2] and derive existence of Hölder continuous weak solutions using also the time discretization. Furthermore, in a two-dimensional setting they show short-time existence and $C^{1,1}$-regularity of solutions. Finally, they apply their results to the problem of positive curvature flow in two dimensions and relate that problem to a biological model. Spadaro also employes the time discretization in [120] to consider mean curvature flow with obstacles. He characterizes the least barrier enclosing all minimal hypersurfaces with boundary on a given set as the asymptotic limit of such a flow. Following and extending the classical work of Ecker and Huisken (see [43]) and using the penalization method, Mercier and Novaga [103] generalize the result of [2] to higher dimensions. They also consider the case of periodic and entire graphs. In subsequent work Mercier generalizes the level-set approach to mean curvature flow with obstacles [102]. Finally, in [116], Rupflin and Schnürer treat (parametric) mean curvature flow with obstacles via the approach of Sáez and Schnürer [117] to mean curvature flow. Roughly speaking, this approach lifts the evolution problem to one dimension higher via constructing complete graphs over the initial surface. One is then lead to the problem of studying the (simpler) evolution of those graphical solutions with changing domains – which correspond to the evolution of the original set. Generalizing this approach to the case of mean curvature flow with obstacle Rupflin and Schnürer are lead to consider an obstacle problem for complete graphs with time-dependent domain. Using also the penalization method together with an extension of the estimates from [43], they are able to show existence and optimal regularity in the case of complete graphs and – coming back to the problem they originally considered – propose another weak notion of mean curvature flow with obstacles. In part C of the appendix we comment on the method of penalization and show how it could be applied to our Dirichlet problem.

Open Problems

We address now a couple of open questions and possible further research directions in connection with this problem and refer again to the relevant literature. First of all, there is an abundance of questions related to the study of the regularity and structure of the free boundary. The interest in these kind of questions started with the examples of Schaeffer [118], exhibiting first examples of singularities in free boundaries. Caffarelli then introduced the *blow-up* method to the study of obstacle problems in his seminal papers [22, 24], thus creating a strong link between obstacle problems and geometric measure theory, where blow-up methods were already successfully applied. Kinderlehrer considered the case of free boundaries in constrained minimal surfaces and proved their analyticity under suitable assumptions on the obstacle in [86]. Caffarelli's approach was later simplified by Weiss' celebrated epiperimetric and monotonicity formula [125] and by Monneau in [108]. Important contributions to the parabolic case where initiated again by Weiss' approach [126] as well as by Caffarelli-Petrosyan-Shahgholian [21] and Shahgholian-Uraltseva-Weiss [119]. We would like to also mention the recent advances on the study of the singular set (of the free boundary) in parabolic obstacle problems by Blanchet [16] and Lindgren-Monneau [94]. It would also be interesting to answer the question, if the variational inequality (0.2) is strong enough to deduce uniqueness of distributional solutions. Furthermore, also a proof for the optimal regularity that avoids the use of the penalization method would be worthwhile. Analogous to [3], one could also try to consider other kinds of curvature flows. Also with respect to the obstacle, one could study some generalizations. Apart from considering time dependent obstacles one could also investigate if the scheme allows to consider so called *thin obstacles*. These are obstacles, that just live on the intersection of the domain with a lower dimensional hyperplane. Among the many contributions to this particular type of obstacle problem, we can also only mention few. The $C^{1,\alpha}$ regularity was shown by Caffarelli [23] and, using another method, by Uraltseva [123]. Only rather recently, the optimal regularity (in the Laplace-case) was shown by Athanasopoulos-Caffarelli [5]. A different proof for this result, using the so called *frequency function* introduced by Almgren, was given in Athanasopoulos-Caffarelli-Salsa [6] which also paved the way and initiated a lot of work on the regularity of the free boundary, we only mention Garofalo-Petrosyan [58]. Finally, we would also like to mention that in the last couple of years a number of publications are starting to investigate the thin obstacle problem in the variable-coefficient case. Here we only refer to Garofalo-Smit Vega Garcia [60], Garofalo-Petrosyan-Smit Vega Garcia [59], Banerjee-Smit Vega Garcia-Zeller [8] (for the parabolic thin obstacle), Focardi-Spadaro [50] and Koch-Rüland-Shi [88], [89]. For a more complete overview of the richness of the thin obstacle problem we refer also to the ninth chapter of the monograph [112].

Contents of the thesis

Chapter one is of introductory and expository character and starts with recalling the classical time discretization scheme by Almgren-Taylor-Wang and Luckhaus-Sturzenhecker. After a short interlude on signed distance functions the scheme will be adapted according to the necessities of our parabolic obstacle problem. The remainder of the chapter is dedicated to a first analysis of this adapted scheme. We prove existence and graphicality of minimizers of the elliptic problems and motivate a reformulation of the scheme in the class of Lipschitz functions. Finally we show that this new formulation yields the same solutions.

The study of a class of obstacle problems, which encompasses the minimization problems arising from the reformulated scheme is then the content of the second chapter. We are employing an approach introduced by Hartman and Stampacchia [75] which transforms these problems – which are instances of (elliptic) *variational inequalities* – to a problem on an even smaller class of functions and eventually reduces to the construction of so called *barriers*, a tool to deduce a priori boundary gradient estimates. Although this is just one possible way of deducing the existence of Lipschitz solutions of the time-independent elliptic problems – another one is also discussed – this one allows us to derive suitable bounds on the Lipschitz constant (Theorem 2.2.8). This argument can then be iterated in the next chapter. We will then also discuss the higher regularity of solutions to certain obstacle problems. Namely, we deduce the $W^{2,p}(\Omega)$ and $C^{1,1}$-estimates, without using the usual penalization arguments. Eventually, we use the results from this chapter to derive existence, uniqueness and regularity of constrained minimal graphs together with a useful characterization of them.

Chapter three begins with a detailed explanation how the results of the second chapter can be used to derive the existence of *approximate flows* with uniform control on the (spacial-) Lipschitz constant (Theorem 3.1.3). Subsequently, this Lipschitz bound is heavily used to derive a number of important properties of this time discrete evolution. These results can then be used to pass into the limit in the time step size $h \to 0$ to obtain so called *flat flows*. After deriving L^2-spacetime bounds on the derivatives of these flat flows, we show that they are indeed solving the variational inequality (0.2) and can thus also be called *distributional solutions* (Theorem 3.4.8). A final section deals with the asymptotic limit as $t \to +\infty$. We show that flat flows converge uniformly to constrained minimal graphs (Proposition 3.5.1).

In the final chapter, we first of all introduce yet another way to formulate our parabolic obstacle problem, namely the notion of *viscosity solutions*. This concept is particularly flexible as it requires the solutions merely to be continuous. Then we show that the flat flows obtained in chapter three are also viscosity solutions (Proposition 4.2.2). Having established this connection, we deduce a variant of the classical comparison principle for viscosity sub- and supersolutions of the parabolic obstacle problem (Proposition 4.3.1) which not only gives us another characterization of flat flows but also a way to deduce uniqueness of flat flows and thus to remove the requirement to pass to subsequences of approximate flows (Corollary 4.4.2).

The appendix consists of three parts. In the first one, we collect regularity results for quasi-linear elliptic PDE's that we employ in this thesis. We provide either a reference or a sketch of the proofs. Part B of the appendix is a collection of results from different fields, starting from properties of the mean curvature operator, Caccioppoli sets, including a basic auxiliary results on families of equicontinuous functions and finally presenting a proof of a useful result related to the concept of semijets. The final part is dedicated to the so called penalization-approach on obstacle problems and is intended to give the reader a flavor of the arguments and computations involved in this strategy.

1. The Time Discretization Scheme

In this first chapter we give a thorough introduction to the approach we choose to treat our parabolic obstacle problem and which will help us to overcome the difficulties discussed in the introduction.

In a first section we quickly recall the time discretization scheme for mean curvature flow as introduced by Almgren-Taylor-Wang [3] and Luckhaus-Sturzenhecker [96]. Before we then adapt the scheme to our problem we will collect some basic properties about distance functions which will be used throughout the whole thesis. Subsequently we introduce the non-parametric obstacle problem via an analog of the classical scheme and some first results towards existence, uniqueness and regularity of (approximate) flows are derived.

1.1. The Time Discretization for Mean Curvature Flow

As we discussed already in the introduction, the development of singularities in mean curvature flow has led to a variety of so called weak formulations, which allow to investigate the evolution of the surface even past a singularity. Our interest is now focused on one of those weak formulations which was developed independently by Almgren-Taylor-Wang [3] and Luckhaus-Sturzenhecker [96]. Let us start by giving a very rough overview of the scheme. The main idea is to consider the evolving (n-dimensional) surface as the boundary of an evolving ($(n+1)$-dimensional) subset of $\mathbb{R}^{n+1}$. The evolution of this set is then obtained as a limit of so called *approximate flows* which in turn are discrete in time and defined iteratively as solutions of minimization problems.

Let us now be a bit more precise and explain how this scheme is implemented. Typically, the initial surface is given by a compact, smooth, embedded, n-dimensional submanifold M_0 of $\mathbb{R}^{n+1}$. Instead we start with an initial set $E_0 \subset \mathbb{R}^{n+1}$ which we require to be of finite perimeter and we think of ∂E_0 as the replacement for M_0. For more about sets of finite perimeter we refer to the monograph [97]. We just recall that $E \subset \mathbb{R}^{n+1}$ is called a *set of finite perimeter* (or equivalently a *Caccioppoli set*) if

$$\operatorname{Per}(E) := \sup\left\{ \int_E \operatorname{div}\varphi \,\mathrm{d}x \,:\, \varphi \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1}),\ \sup_{x\in\mathbb{R}^{n+1}} |\varphi(x)| \le 1 \right\} < +\infty,$$

and we call $\operatorname{Per}(E)$ the *perimeter* of E. For a fixed time step size $h > 0$ we set $E^h_0 := E_0$ and we define E^h_k iteratively for each $k \in \mathbb{N}$ as being a minimizer of

$$E \mapsto \operatorname{Per}(E) + \frac{1}{h} \int_{E^h_{k-1}\Delta E} \operatorname{dist}_{\partial E^h_{k-1}}(x)\,\mathrm{d}x, \tag{1.1}$$

where E varies among all Caccioppoli sets.

Remark 1.1.1. Let us recall that usually one identifies sets of finite perimeter who agree almost everywhere. Thus, the definition of the distance function above requires some care. To be precise, either we could choose a suitable representative of E, or more directly (but by slight abuse of notation) we could set

$$\mathrm{dist}_{\partial E}(x) := \inf_{y \in \tilde{\partial} E} |x-y|,$$

where $\tilde{\partial} E := \{x \in \mathbb{R}^{n+1} : 0 < \left|E \cap B_r(x;\mathbb{R}^{n+1})\right| < \omega_{n+1} r^{n+1} \ \forall r > 0\}$. Here and in the following $|A|$ (where $A \subset \mathbb{R}^m$) refers to the m-dimensional Lebsesgue measure and ω_m is the volume of the unit ball in $\mathbb{R}^m$. It is then immediate to check that whenever $|E \Delta F| = 0$ we have $\tilde{\partial} E = \tilde{\partial} F$ which shows well-definedness. However, apart from this introductory first section the sets to which we compute distances will always be pointwise defined and this subtlety will not matter.

Given such a sequence $(E_k^h)_{k \in \mathbb{N}}$ as above, we then define as an *approximate flow* the family $(E_t^h)_{t \geq 0}$ defined by interpolation as

$$E_t^h := E_k^h, \quad \text{if } t \in [kh, (k+1)h[.$$

Finally, we define a family of sets of finite perimeter $(E_t)_{t \geq 0}$ as *weak mean curvature flow* or – in the terminology of [3] – as *flat flow*, if there exists a sequence $h_k \to 0$ as $(k \to \infty)$ and a sequence of approximate flows $(E_t^{h_k})$ with the property that for every $t \geq 0$, $\chi_{E_t^{h_k}} \to \chi_{E_t}$ in $L^1(\mathbb{R}^{n+1})$.

Before we discuss the known results we would like to give some more intuition about the functional (1.1). First of all it is important to understand the antagonistic relation between the perimeter-term and the bulk term. While the first one tries to minimize the perimeter in each step of the iteration, the second one will penalize in some way how much the new set will deviate from the old one. We would like to dwell a bit more on an important feature of this second term. As we are dealing with sets of finite perimeter, probably the most natural way of measuring the distance between E and E_{k-1}^h would be

$$\left|E \Delta E_{k-1}^h\right| = \int_{E_{k-1}^h \Delta E} 1 \, \mathrm{d}x.$$

However, using this distance to measure the deviation of E from E_{k-1}^h would lead to some kind of non-local evolution as it would be more favorable to entirely concentrate the variation in volume to that part of ∂E_{k-1}^h with highest curvature, for details we refer to section 3.1 in [4]. Following the heuristics given in [3] (cf. section 2.12 therein) we would like to convince the reader why using the distance to the boundary is the right choice to recover evolution by mean curvature. Recalling that a family of hypersurfaces M_t is evolving by mean curvature flow, if, for every time and every point on the surface, its normal velocity equals the mean curvature (vector) at this point, i.e.

$$\partial_t x = \vec{H}_{M_t}(x).$$

At regular points of ∂E_k^h, as we will see later, the equation

$$\left|\vec{H}_{\partial E_k^h}(x)\right| = \frac{1}{h}\mathrm{dist}_{E_{k-1}^h}(x),$$

will hold. Interpreting E_{k-1}^h and E_k^h as the evolving surface at times $(k-1)h$ and kh a natural candidate for the discretized speed in normal direction is therefore given by the right hand side of the previous equation.

The main result that is proved in both [3] and [96] is a kind of (conditional) existence and (short time) consistency statement. Namely, they prove that if E_0 has smooth boundary, then there exists a weak mean curvature flow (or flat flow) and every such flow coincides with the evolution of ∂E_0 according to mean curvature flow up to the time when the first singularity appears.

Finally, we would like to mention that this method can seen as a special case of so called *minimizing movements*, as introduced by DeGiorgi in [39]. The interested reader who wants to learn more about this notion and also about its relations to the notion of gradient flows in metric spaces which gained a lot of attention in recent years is referred to [4].

1.2. Definition and Basic Properties of the Signed Distance Function

As we already saw in the first section, the distance function will play a crucial role in the discretization scheme. Moreover, we will also need to understand the relation between the distance function and the signed distance function and finally, due to the fact that we also want to deal with Dirichlet boundary conditions, we need to carefully analyze the influence of the underlying domain (which in contrast to the classical scheme will no longer be $\mathbb{R}^{n+1}$ but only a subset of $\mathbb{R}^{n+1}$). Hence, this section should systematically collect all the results (and notations) which will be needed subsequently. We start by recalling the definition of the (signed) distance function and some elementary properties. In the following let $A \subset V \subset \mathbb{R}^m$ for some $m \in \mathbb{N}$.

Definition 1.2.1. We define the distance to A (in V) as

$$\begin{aligned} \mathrm{dist}_A : V &\to [0, +\infty], \\ x &\mapsto \inf_{a\in A} |x-a|, \end{aligned}$$

and the signed distance to A (in V) as

$$\begin{aligned} \mathrm{sdist}_A : V &\to \mathbb{R}, \\ x &\mapsto \mathrm{dist}_A(x) - \mathrm{dist}_{V\setminus A}(x) = \begin{cases} \mathrm{dist}_A(x) & \text{if } x \notin A, \\ -\mathrm{dist}_{V\setminus A}(x) & \text{if } x \in A. \end{cases} \end{aligned}$$

Note, that the sign convention on the signed distance function is not coherent in the literature. Here, we choose the version which suits our purpose best and which will make sdist_A coincide with the distance function outside of A. Moreover, if we want to emphasize which is the underlying space, we write $\mathrm{dist}_A(\cdot; V)$ and $\mathrm{sdist}_A(\cdot; V)$. The following monotonicity property is immediate.

Lemma 1.2.1. *Let A and B be two subsets of $\mathbb{R}^m$ with $A \subset B$. Then for every $x \in \mathbb{R}^m$ we have*

$$\mathrm{dist}_B(x) \leq \mathrm{dist}_A(x) \quad \textit{and} \quad \mathrm{sdist}_B(x) \leq \mathrm{sdist}_A(x).$$

Proof. By hypothesis we know that $\{|x-y| : \ y \in A\} \subset \{|x-y| : \ y \in B\} \subset \mathbb{R}$. Taking the infimum yields the first claim. The statement for the signed distance follows now by a simple case analysis. □

The following example illustrates the effects of changing the underlying space.

Example 1.2.1. Let $V_1 = [0, +\infty[$ and let $A = [0, 1]$, endowed with the usual metric. For $x \in V_1$ we have $\mathrm{sdist}_A(x; V_1) = x - 1$. Instead, if we let $V_2 = \mathbb{R}$, then for $x \in V_2$ we have $\mathrm{sdist}_A(x; V_2) = \left|x - \frac{1}{2}\right| - \frac{1}{2}$. Note, in particular, that at the same time $\mathrm{dist}_A(\cdot, V_2)$ will just be an extension of $\mathrm{dist}_A(\cdot, V_1)$ to $\mathbb{R}$.

Nevertheless, in our case, the underlying space will always be fixed and hence we will often not keep track of it in the notation.

We are also interested in the regularity of the distance and signed distance functions. As seen already in the simple example of the distance to a point, in general we can at most hope for Lipschitz regularity. For the distance function, it is quite easy to check that this is also the least we can expect.

Proposition 1.2.2. *Let $V \subset \mathbb{R}^m$ and $\emptyset \neq A \subset V$. Then, dist_A is $1-$Lipschitz.*

Proof. Let $x, y \in V$. By the definition of dist_A and the triangle inequality we get $\mathrm{dist}_A(x) \leq |x - z| \leq |x - y| + |y - z|$ for any $z \in A$. Taking the infimum with respect to z we deduce $\mathrm{dist}_A(x) \leq |x - y| + \mathrm{dist}_A(y)$. Exchanging the roles of x and y allows us to conclude $|\mathrm{dist}_A(x) - \mathrm{dist}_A(y)| \leq |x - y|$. □

Let us now discuss the regularity of the signed distance function. By definition, the signed distance is the difference of two distance functions, and more precisely, according to the previous proposition, it is the difference of two 1-Lipschitz function. Hence the signed distance is at least 2-Lipschitz. The following example shows, that in general this is all we can say about signed distances. Although, for simplicity, the following example takes place in a metric setting – and thus Definition 1.2.1 needs to be generalized in the obvious way – it is not difficult to find a signed distance function with Lipschitz constant strictly bigger than one in the Euclidean setting.

Example 1.2.2. Let (X, d) be the real line endowed with the discrete metric and Consider $A = [-1, 1]$. Then we get $\mathrm{dist}_A = \chi_{\mathbb{R}\setminus[-1,1]}$ and for the signed distance: $\mathrm{sdist}_A = \chi_{\mathbb{R}\setminus[-1,1]} - \chi_{]-1,1[}$. Hence we see that $\mathrm{sdist}_A(2) - \mathrm{sdist}_A(0) = 2 = 2d(0, 2)$.

However, if we restrict to settings in which the underlying space is a convex subset of $\mathbb{R}^m$, then the signed distance will also be 1-Lipschitz.

Proposition 1.2.3. *Let $V \subset \mathbb{R}^m$ be convex and $\emptyset \neq A \subset V$. Then , sdist_A, the signed distance to A in V is 1-Lipschitz.*

Proof. We start by noticing that whenever either both x and y belong to A or both belong to $V \setminus A$ the estimate $|\mathrm{sdist}_A(x) - \mathrm{sdist}_A(y)| \leq |x - y|$ follows as seen in the proof of Proposition 1.2.2. In order to discuss the remaining cases, we can assume without loss of generality that $x \in A$ and $y \in V \setminus A$. Moreover, we find some point $z \in \partial A \cap [x; y]$. Notice, that by this choice we have $|x - y| = |x - z| + |z - y|$ and $\mathrm{sdist}_A(z) = 0$. Consequently, we get $|\mathrm{sdist}_A(x) - \mathrm{sdist}_A(y)| \leq |\mathrm{sdist}_A(x) - \mathrm{sdist}_A(z)| + |\mathrm{sdist}_A(z) - \mathrm{sdist}_A(y)|$. As both x and z have distance 0 to A we can use again the previous proposition applied to $\mathrm{dist}_{V \setminus A}$ to estimate the first term by $|x - z|$. Analogously, one can estimate the second term by $|z - y|$. This allows us to conclude that sdist_A is 1-Lipschitz. □

The following simple lemma about the gluing of Lipschitz functions will be used in several instances in this thesis. We mention it here only because it gives another criterion when sdist_A is 1-Lipschitz.

Lemma 1.2.4. *Let $A \subset V \subset \mathbb{R}^m$ and suppose $f \in \mathrm{Lip}(A)$, $g \in \mathrm{Lip}(V \setminus A)$ such that $f_{|\partial A} = g_{|\partial A}$. Then, also $h : V \to \mathbb{R}$, defined via*

$$h(x) := \begin{cases} f(x) & x \in A, \\ g(x) & x \in V \setminus A, \end{cases}$$

is Lipschitz continuous. Moreover, if both f and g are L-Lipschitz, then so is h.

Proof. Note, that already in the statement we made use of the convention that Lipschitz functions are always assumed to be defined up to the boundary. Of course, it suffices to check that if both f, g are L-Lipschitz, $x \in A$, $y \in V \setminus A$ then

$$|h(x) - h(y)| \leq L|x - y|.$$

Observe, that for ever such pair of points x, y the segment $[x; y]$ has nonempty intersection with ∂A. Let therefore $z \in \partial A \cap [x; y]$ and we get

$$h(x) - h(y) = f(x) - \underbrace{f(z) + g(z)}_{=0} - g(y) \leq L(|x - z| + |z - y|) = L|x - y|.$$

Since we get analogously also $h(y) - h(x) \leq L|x - y|$ the lemma is proved. □

As an easy consequence we get the following.

Corollary 1.2.5. *Let $A \subset V \subset \mathbb{R}^m$ with $\partial A \subset V$. Then $\mathrm{sdist}_A(\cdot, V)$ is 1-Lipschitz.*

Proof. Let $f = \mathrm{dist}_{V \setminus A}$ and $g = \mathrm{dist}_A$. Then the claim follows by using the previous lemma and recalling that by Proposition 1.2.2 f and g are 1-Lipschitz. □

The regularity of the signed distance function increases with the regularity of the boundary.

Proposition 1.2.6. *Let $A \subset \mathbb{R}^m$ be an open, bounded set with ∂A C^k-regular, where $k \geq 2$. Then there exists some $\varepsilon = \varepsilon(A) > 0$ such that* sdist_A *is C^k when restricted to the ε-tubular neighborhood of ∂A.*

Proof. See Lemma 14.16 in [68]. □

Most of the times we will only consider distances and signed distances to graphs or subgraphs of functions. Therefore, we introduce the following notation.

Definition 1.2.2. Let $\Omega \subset \mathbb{R}^n$, and suppose $u : \Omega \to \mathbb{R}$ is Lipschitz. Then, for every $(x, z) \in \Omega \times \mathbb{R}$, we set

$$\mathrm{sdist}_u(x,z) := \mathrm{sdist}_E((x,z), \Omega \times \mathbb{R}) \quad \text{and} \quad \mathrm{dist}_u(x,z) := |\mathrm{sdist}_u(x,z)|,$$

where $E = \{(x,z) \in \Omega \times \mathbb{R} \,:\, z < u(x)\}$ is the (open) subgraph of u. In our scheme, there will naturally appear the distance to

$$\mathrm{Gr}(u) := \{(x,z) \in \Omega \times \mathbb{R} \,:\, z = u(x)\},$$

the graph of u. It is straightforward to check that $\mathrm{Gr}(u) = \partial E \cap (\Omega \times \mathbb{R})$, where ∂E denotes the usual boundary of E in $\mathbb{R}^{n+1}$.

The following relation between $\mathrm{dist}_{\mathrm{Gr}(u)}$ and sdist_u will be crucial.

Lemma 1.2.7. *Let $\Omega \subset \mathbb{R}^n$ be convex. Then we have*

$$\mathrm{sdist}_u(x,z) = \begin{cases} -\mathrm{dist}_{\mathrm{Gr}(u)}(x,z) & \text{if } (x,z) \in E, \\ \mathrm{dist}_{\mathrm{Gr}(u)}(x,z) & \text{else,} \end{cases}$$

where, as in the previous definition, E denotes the subgraph of u.

Proof. Suppose first, that $(x,z) \in (\Omega \times \mathbb{R}) \setminus E$. Then, by definition, we have

$$\mathrm{sdist}_u(x,z) = \mathrm{dist}_E(x,z) = \inf_{(\xi,\zeta) \in E} |(x,z) - (\xi,\zeta)|.$$

By density of E in $\bar{E}$, one can then show that $\mathrm{sdist}_u(x,z) = \inf_{(\xi,\zeta)\in\bar{E}} |(x,z) - (\xi,\zeta)|$ so that, since $\mathrm{Gr}(u) \subset \bar{E}$, Lemma 1.2.1 implies that $\mathrm{sdist}_u(x,z) \leq \mathrm{dist}_{\mathrm{Gr}(u)}(x,z)$. On the other hand, we let $(x_0, z_0) \in \bar{E}$ such that $\mathrm{sdist}_u(x,z) = |(x,z) - (x_0,z_0)|$. We have to show that $(x_0, z_0) \in \mathrm{Gr}(u)$. By definition of E, we know that $z_0 \leq u(x_0)$. Let us assume by contradiction that $z_0 < u(x_0)$. Then, either $u(x_0) \leq z$, in which case the contradiction follows by a comparison to the distance of (x,z) to $(x_0, u(x_0)) \in \bar{E}$. Or otherwise we have $u(x_0) > z$ and the contradiction is reached by considering an intermediate point $\hat{x} \in [x_0; x]$ with $u(\hat{x}) = z$ and then computing the distance of (x,z) to the point $(\hat{x}, z) \in \bar{E}$. Hence $(x_0, z_0) \in \mathrm{Gr}(u)$. The case $(x,z) \in E$ follows similarly. □

1.3. The Time Discretization Scheme in the Graphical Setting with Obstacle

Let us now introduce the adaptations we will make in the graphical setting with a fixed obstacle. We start by recalling what we mean by *graphical setting*. First of all, with respect to the general scheme introduced in the last section, our initial surface E_0 will be given as the subgraph of some function $u_0 \in C^2(\overline{\Omega})$, i.e.

$$E_0 = \operatorname{subgraph}(u_0) := \{(x, z) \in \Omega \times \mathbb{R} \;:\; z < u_0(x)\}.$$

Since we are only interested in that part of the boundary of E_0 which coincides with the graph of u_0, we will replace $\operatorname{Per}(E)$ in (1.1) by $\operatorname{Per}(E; \Omega \times \mathbb{R})$. From now on, the domain Ω is an open, bounded and convex subset of $\mathbb{R}^n$ with smooth boundary. Morally, we want to fix the evolving surface along the boundary $\partial\Omega$. More precisely, we impose on the evolving sets E_t^h (which a priori will just be Caccioppoli sets) that they satisfy

$$\operatorname{Tr}(\chi_{E_t^h}) = \operatorname{Tr}(\chi_{E_0}) \quad \text{on } \partial\Omega \times \mathbb{R}.$$

Finally, the obstacle will be given by a function $\psi \in C^{1,1}(\Omega)$ and we require E_t^h to contain the subgraph of ψ for all times $t \geq 0$. In particular, we have to assume that $u_0 \geq \psi$ in Ω and furthermore we always assume that $u_0 > \psi$ on $\partial\Omega$.

Constraining the evolution to always contain the obstacle can be achieved easily by restricting the class of competitors in the obvious way. Namely, by restricting it to those Caccioppoli sets containing the subgraph of ψ. We will see that this constraint is stable in the sense that it will be easy to verify it for minimizers of our adapted functional. As for the boundary condition, we face the problem that a prescribed trace of BV-functions might not pass into the L^1-limit. Therefore, we follow the well known idea of introducing a penalization-term in the functional which will make it energetically favorable to attain the boundary data. At the same time, we will allow also competitors that do not attain the boundary data.

The preceding discussion suggests to consider the following adaption of the original functional (1.1): For a fixed time step size $h > 0$ we consider the problem of minimizing

$$\mathcal{F}_{E_0,h}(E) := \operatorname{Per}(E; \Omega \times \mathbb{R}) + \frac{1}{h} \int_{E_0 \Delta E} \operatorname{dist}_{\operatorname{Gr}(u_0)}(x; \Omega \times \mathbb{R}) \,\mathrm{d}x + \int_{\partial\Omega \times \mathbb{R}} |\chi_E - \chi_{E_0}| \,\mathrm{d}\mathcal{H}^n,$$

among all E in

$$\mathcal{C} := \{E \subset \Omega \times \mathbb{R} \text{ meas.} \,|\, \operatorname{Per}(E; \Omega \times \mathbb{R}) < +\infty,\ \Psi \subset E\},$$

with $\Psi := \operatorname{subgraph}(\psi)$. We would like to emphasize that here and in the following the distance functions (and later the signed distances) are considered as distances from subsets of $\Omega \times \mathbb{R}$ and not in $\mathbb{R}^{n+1}$. Using the notation from the previous section, as a corollary of Lemma 1.2.7 we could also write $\operatorname{dist}_{u_0}(x)$. The geometric idea to keep in mind is that we are computing the distance to the graph of u_0 – which is only a subset of the boundary of the subgraph of u_0. This is similar to the reason why we consider $\operatorname{Per}(E; \Omega \times \mathbb{R})$ instead of the original term $\operatorname{Per}(E)$.

First Existence Result

As in [100] we will show the existence of minimizers of $\mathcal{F}_{E_0,h}$ by considering an equivalent problem which is of so called *mean curvature type*. This reformulation of the problem allows us to get the existence by a simple application of the direct method in combination with the compactness theorem for BV-functions.

We will absorb the third term of our energy by the following classical trick. Fix $\rho > 0$ such that $\overline{\Omega} \subset B_\rho(0) =: \widetilde{\Omega}$ and let $\widetilde{u_0}$ be a C^2-extension of u_0 to $\widetilde{\Omega}$. We denote by $\widetilde{E_0}$ the subgraph of this extension, i.e. $\widetilde{E_0} := \{(x,z) \in \widetilde{\Omega} \times \mathbb{R} \,:\, z < \widetilde{u_0}(x)\}$. Then we consider the new set of competitors

$$\widetilde{\mathcal{C}} := \{E \subset \widetilde{\Omega} \times \mathbb{R} \text{ meas.} \mid \operatorname{Per}(E; \widetilde{\Omega} \times \mathbb{R}) < +\infty,\ \Psi \subset E,\ E \setminus (\Omega \times \mathbb{R}) = \widetilde{E_0} \setminus (\Omega \times \mathbb{R})\},$$

on which we define the functional

$$\widetilde{\mathcal{F}}_{E_0,h}(E) := \operatorname{Per}(E; \widetilde{\Omega} \times \mathbb{R}) + \frac{1}{h} \int_{E \cap (\Omega \times]c,+\infty[)} \operatorname{sdist}_{u_0}(x)\,\mathrm{d}x,$$

for $c := \inf_\Omega \psi$. Without loss of generality we henceforth assume that $c = 0$.

Proposition 1.3.1. *There exists some $\widetilde{E} \in \widetilde{\mathcal{C}}$ which minimizes $\widetilde{\mathcal{F}}_{E_0,h}$.*

Proof. First of all, let us check that $\widetilde{\mathcal{F}}_{E_0,h}$ is bounded from below on $\widetilde{\mathcal{C}}$. Since the perimeter is non-negative, we only need to bound the volume term. As $\operatorname{sdist}_{u_0} \geq 0$ on $(\Omega \times \mathbb{R}) \setminus E_0$ and $\operatorname{sdist}_{u_0} \leq 0$ on E_0 we can estimate

$$\frac{1}{h} \int_{E \cap (\Omega \times [0,+\infty[)} \operatorname{sdist}_{u_0}(x)\,\mathrm{d}x \geq \frac{1}{h} \int_{E_0 \cap (\Omega \times [0,+\infty[)} \operatorname{sdist}_{u_0}(x)\,\mathrm{d}x > -\infty.$$

Noting that the second integral is not depending on E anymore, we found a lower bound for $\widetilde{\mathcal{F}}_{E_0,h}$. Let now $(E_k)_{k\in\mathbb{N}}$ be a minimizing sequence in $\widetilde{\mathcal{C}}$, i.e. a sequence such that

$$\widetilde{\mathcal{F}}_{E_0,h}(E_k) \to \inf_{E \in \widetilde{\mathcal{C}}} \widetilde{\mathcal{F}}_{E_0,h}(E) \quad (k \to +\infty).$$

Since $\widetilde{\mathcal{F}}_{E_0,h}$ is bounded from below, this infimum is finite. Moreover, by the boundedness of the volume-term we deduce that

$$\sup_{k\in\mathbb{N}} \operatorname{Per}(E_k; \widetilde{\Omega} \times \mathbb{R}) < +\infty.$$

Without loss of generality we can also assume that there is some $R > 0$ such that $E_k \subset \widetilde{\Omega} \times]-\infty, R[$, for every $k \in \mathbb{N}$. If this were not the case, we could take $R > 0$ such that $\widetilde{E_0} \subset \widetilde{\Omega} \times]-\infty, R[$. Now it is not hard to see that $E_{k,R} := E_k \cap (\widetilde{\Omega} \times]-\infty, R[)$ (which also belongs to $\widetilde{\mathcal{C}}$) is performing at least as good as E_k. Indeed, the perimeter of $E_{k,R}$ is less or equal than the one of E_k since intersecting with convex sets always reduces the perimeter. Regarding the volume term, one only needs to observe that by our choice of R, on $\Omega \times]R, +\infty[$ we have $\operatorname{sdist}_{u_0} \geq 0$. Thus we get

$$\int_{E_k \cap (\Omega \times [0,+\infty[)} \operatorname{sdist}_{u_0}(x)\,\mathrm{d}x \geq \int_{E_{k,R} \cap (\Omega \times [0,+\infty[)} \operatorname{sdist}_{u_0}(x)\,\mathrm{d}x.$$

Consequently, we showed that one can construct a bounded (minimizing) sequence $(E_k)_{k\in\mathbb{N}}$ with equibounded perimeters. Hence, we can apply a variant of the well-known compactness result for sets of finite perimeter (see [97, Theorem 12.26]). More precisely, we will apply the compactness theorem to the sets

$$F_k := E_k \cap (\Omega\times]0, R[).$$

By the compactness theorem we get that a subsequence of $(F_k)_{k\in\mathbb{N}}$ is converging to a set of finite perimeter, say F, in $L^1(\mathbb{R}^{n+1})$. Noting that

$$E_k \setminus F_k = (\widetilde{E_0} \setminus (\Omega \times \mathbb{R})) \cup \Psi \quad \forall k \in \mathbb{N},$$

we see that (up to possibly passing to a subsequence) E_k converges in $L^1(\mathbb{R}^{n+1})$ to $\widetilde{E} := F \cup ((\widetilde{E_0} \setminus (\Omega \times \mathbb{R})) \cup \Psi)$.

Note, that $\widetilde{\mathcal{F}}_{E_0,h}$ is lower-semi-continuous with respect to L^1-convergence. In fact, the perimeter term is lower-semi-continuous (cf. [97, Proposition 12.15]) and the volume term is even continuous with respect to L^1-convergence. This gives us finally the estimate

$$\widetilde{\mathcal{F}}_{E_0,h}(\widetilde{E}) \leq \liminf_{k\to\infty} \widetilde{\mathcal{F}}_{E_0,h}(E_k) = \inf_{E\in\widetilde{\mathcal{C}}} \widetilde{\mathcal{F}}_{E_0,h}(E).$$

After possibly passing to another subsequence, we can also assume that the convergence is point-wise almost everywhere and we get $\widetilde{E} \subset \widetilde{\Omega} \times \mathbb{R}$, $\Psi \subset \widetilde{E}$ and $\widetilde{E} \setminus (\Omega \times \mathbb{R}) = \widetilde{E_0} \setminus (\Omega \times \mathbb{R})$, i.e. $\widetilde{E} \in \widetilde{\mathcal{C}}$ which allows us to conclude that $\widetilde{E}$ minimizes $\widetilde{\mathcal{F}}_{E_0,h}$ in $\widetilde{\mathcal{C}}$. □

Remark 1.3.1. Note that $R = \|\widetilde{u_0}\|_{L^\infty(\widetilde{\Omega})}$ does the job, so this proof actually gives an L^∞-bound for $\widetilde{E}$.

As a simple consequence of the preceding proposition we get the following existence result.

Proposition 1.3.2. *There exists a minimizer of $\mathcal{F}_{E_0,h}$ in $\mathcal{C}$.*

Proof. Given a set $E \in \mathcal{C}$, by setting $\widetilde{E} := E \cup (\widetilde{E_0} \setminus (\Omega \times \mathbb{R}))$ we obtain a competitor in $\widetilde{\mathcal{C}}$. Vice versa, for any $\widetilde{E}$ in $\widetilde{\mathcal{C}}$ we obtain a set belonging to $\mathcal{C}$ by setting $E := \widetilde{E} \cap (\Omega \times \mathbb{R})$. Moreover, by a variant of Theorem 16.16 in [97] we get that for any $\widetilde{E} \in \widetilde{\mathcal{C}}$ we have

$$\operatorname{Per}(\widetilde{E}; \widetilde{\Omega} \times \mathbb{R}) = \operatorname{Per}(\widetilde{E_0}; (\widetilde{\Omega} \setminus \overline{\Omega}) \times \mathbb{R}) + \operatorname{Per}(E; \Omega \times \mathbb{R}) + \int_{\partial\Omega\times\mathbb{R}} |\chi_E - \chi_{E_0}| \, \mathrm{d}\mathcal{H}^n. \quad (1.2)$$

Note next, that for every $E \in \mathcal{C}$ we have $E\Delta E_0 \subset (\Omega\times]0, +\infty[) =: A$ (We recall that without loss of generality $\inf_\Omega \psi = 0$). Consequently, by Lemma 1.2.7 and the definition of dist_{u_0} we get

$$\begin{aligned}\int_{E\Delta E_0} \mathrm{dist}_{u_0} \, \mathrm{d}x &= \int_{(E\setminus E_0)\cap A} \mathrm{sdist}_{u_0} \, \mathrm{d}x - \int_{(E_0\setminus E)\cap A} \mathrm{sdist}_{u_0} \, \mathrm{d}x \\ &= \int_{E\cap A} \mathrm{sdist}_{u_0} \, \mathrm{d}x - \int_{E_0\cap A} \mathrm{sdist}_{u_0} \, \mathrm{d}x,\end{aligned} \quad (1.3)$$

where we were adding and subtracting $\int_{E\cap E_0\cap A}\mathrm{sdist}_{u_0}$ in the second equality. Combining (1.2) and (1.3) we see that the two functionals $\mathcal{F}_{E_0,h}$ and $\widetilde{\mathcal{F}}_{E_0,h}$ only differ by a constant, or more precisely that for any $\widetilde{E}\in\widetilde{\mathcal{C}}$

$$\widetilde{\mathcal{F}}_{E_0,h}(\widetilde{E}) = \mathcal{F}_{E_0,h}(\widetilde{E}\cap(\Omega\times\mathbb{R})) + \mathit{const}.$$

This allows us to deduce that given a minimizer $\widetilde{E}$ of $\widetilde{\mathcal{F}}$ – which exists by the preceding proposition – $E=\widetilde{E}\cap(\Omega\times\mathbb{R})$ is a minimizer of $\mathcal{F}_{E_0,h}$. □

Remark 1.3.2. As a consequence of computation (1.3) we also see that instead of considering the functional $\mathcal{F}_{E_0,h}$ we could slightly change the volume part and study instead the minimizers of

$$\mathcal{G}_{E_0,h}(E) = \mathrm{Per}(E;\Omega\times\mathbb{R}) + \frac{1}{h}\int_{E\cap A}\mathrm{sdist}_{u_0}(x)\,\mathrm{d}x + \int_{\partial\Omega\times\mathbb{R}}|\chi_E-\chi_{E_0}|\,\mathrm{d}\mathcal{H}^n.$$

As $\mathcal{G}_{E_0,h}$ and $\mathcal{F}_{E_0,h}$ only differ by the constant term $\int_{E_0\cap A}\mathrm{sdist}_{u_0}$, we would get the same minimizers.

Graphicality and Boundary Behavior of Minimizers

The aim of the next paragraph is to establish the important result that the minimizers of the functional $\mathcal{F}_{E_0,h}$ will actually be subgraphs of functions of bounded variation. To begin with, we will however prove that those minimizers attain the boundary data. We will essentially adapt the arguments of section 3 in [100] to our graphical setting.

Proposition 1.3.3. *Let $E\in\mathcal{C}$ be a minimizer of $\mathcal{F}_{E_0,h}$. Then $\chi_E=\chi_{E_0}$ on $\partial\Omega\times\mathbb{R}$ in the sense of traces.*

Proof. It suffices to show that for every $x\in\partial\Omega$ and every $z>u_0(x)$ we have

$$\exists r_0=r_0(x,z)>0: \quad B_{r_0}((x,z);\mathbb{R}^{n+1})\cap E=\emptyset, \tag{1.4}$$

and for every $x\in\partial\Omega$ and every $\psi(x)<z<u_0(x)$ we have

$$\exists r_0=r_0(x,z)>0: \quad B_{r_0}((x,z);\mathbb{R}^{n+1})\cap(\Omega\times\mathbb{R})\subset E. \tag{1.5}$$

Indeed, let us recall that if we denote the trace of χ_E on $\partial\Omega\times\mathbb{R}$ by φ, then for $\mathcal{H}^n$-a.e. $(x,z)\in\partial\Omega\times\mathbb{R}$ we have (cf. Thm. 2.10 in [72])

$$\lim_{r\to0}\left(\frac{1}{r^{n+1}}\int_{B_r((x,z))\cap(\Omega\times\mathbb{R})}|\chi_E(y)-\varphi(x,z)|\,\mathrm{d}y\right)=0.$$

Noting that for (x,z) as in (1.4) we have $\chi_E=0$ on $B_r((x,z))\cap(\Omega\times\mathbb{R})$ whenever $r<r_0(x,z)$, we therefore deduce that

$$|\varphi(x,z)| = \lim_{r\to0}\frac{|B_r((x,z))\cap(\Omega\times\mathbb{R})|}{r^{n+1}}=0.$$

Since $\partial\Omega$ is regular (and has hence no cusps), the above limit is positive and we deduce $\varphi(x,z) = 0$. Similarly, for (x,z) as in (1.5) we would deduce by the same argument that $\varphi(x,z) = 1$. Moreover, we note that (1.5) follows from (1.4) by considering an appropriate mirrored minimization problem (with an obstacle above). Let us therefore derive (1.4). We fix some (x,z) such that $x \in \partial\Omega$ and $z > u_0(x)$ and assume by contradiction that for every $r > 0$ we have

$$B_r((x,z);\mathbb{R}^{n+1}) \cap E \neq \emptyset,$$

i.e. that $(x,z) \in \partial E$, with ∂E denoting the boundary of E as a subset of $\mathbb{R}^n$. For the moment, let us assume that near (x,z) one can write ∂E as the graph of a C^2-function f above $T_{(x,z)}(\partial\Omega \times \mathbb{R}) \cong \mathbb{R}^n$, the tangent plane to $\partial\Omega \times \mathbb{R}$ at (x,z). Moreover, with respect to the same hyperplane, we can also parametrize $\partial\Omega \times \mathbb{R}$ by some C^2-function, say g. By the convexity of Ω we get that

$$\operatorname{div}\left(\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}\right) \geq 0,$$

while at the same time, using the Euler-Lagrange equation for u, we have

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = -\frac{1}{h}\operatorname{sdist}_{E_0}(\cdot, f(\cdot)) < 0.$$

At the same time, by the strong maximum principle (Section 2 in [107]), from the relations

$$\begin{aligned} f(x) &= g(x), \\ f &\geq g, \end{aligned}$$

we infer that $f = g$ and hence the contradiction

$$0 \leq \operatorname{div}\left(\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}\right) = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) < 0.$$

It remains to justify why we could assume that ∂E can be written as a graph near (x,z). To do so, we use a blow-up argument. For $k \in \mathbb{N}$, we set

$$E_k := k(E - (x,z)) := \{(\xi,\zeta) \in \mathbb{R}^n \times \mathbb{R} : \frac{1}{k}(\xi,\zeta) + (x,z) \in E\}.$$

In the language of [97], E is a (Λ, r)-perimeter minimizers in $B := B_{r_0}((x,z);\mathbb{R}^{n+1})$ with $\Lambda = \frac{1}{h}\|\operatorname{sdist}_{u_0}\|_{L^\infty(B)}$, for any $r_0 < z - u_0(x)$ and any $r > 0$. Consequently, E_k are $(\frac{\Lambda}{k}, r)$-perimeter minimizers in $B := B_{kr_0}(0;\mathbb{R}^{n+1})$. Then, by compactness [97, Proposition 21.13], up to passing to a subsequence, we can assume that $E_k \to E_\infty$ in $L^1_{loc}(\mathbb{R}^{n+1})$ as $(k \to \infty)$, where E_∞ is a set of finite perimeter. As E_k was a blow up sequence of a (Λ, r)-perimeter minimizer, we can even deduce that E_∞ is a perimeter minimizing cone with vertex at 0. Moreover, E_∞ is contained in the halfspace $H = \{(\xi, \zeta) \in \mathbb{R}^{n+1}\,;\, (\xi,\zeta)\cdot \nu_(x,z) > 0\}$, where ν is the inward pointing unit normal along $\partial\Omega \times \mathbb{R}$. Hence, by a Bernstein-type theorem (cf. Theorem 15.5 in [72]) we get that $E_\infty = H$. In particular $\partial^* E_\infty = \partial E$ and thus $0 \in \partial^* E_\infty$. This implies now that also $(x,z) \in \partial^* E$ so that by regularity theory ([97, Theorem 26.5] and Theorem A.5), we know that near (x,z), ∂E can be written as a $C^{2,\alpha}$-graph. □

In order to show that minimizers are subgraphs (of BV), we will use a symmetrization-argument for which we introduce the following notation.

Definition 1.3.1. For $\widetilde{E} \in \widetilde{\mathcal{C}}$ we set

$$\mathcal{S}(\widetilde{E}) := \text{subgraph}(f_{\widetilde{E}}),$$

where $f_{\widetilde{E}}(x) := \mathcal{H}^1(\widetilde{E} \cap (\{x\} \times [0, +\infty[))$ for almost every $x \in \widetilde{\Omega}$.

Note, that $f_{\widetilde{E}}$ is well-defined. Indeed, suppose that $\widetilde{F} = \widetilde{E}$ $\mathcal{L}^{n+1}$-almost everywhere. Then, for almost every $x \in \widetilde{\Omega}$, we have $\widetilde{E} \cap (\{x\} \times \mathbb{R}) = \widetilde{F} \cap (\{x\} \times \mathbb{R})$ $\mathcal{L}^1$-almost everywhere. We recall the following properties of $f_{\widetilde{E}}$ and $\mathcal{S}(\widetilde{E})$ which are standard and can be found for instance in [97, Theorem 14.4] and [74, Lemma 1.3.2].

Lemma 1.3.4. *Let $\widetilde{E}$, $\mathcal{S}(\widetilde{E})$ and $f_{\widetilde{E}}$ be as above, then the following holds.*

i) $\mathcal{S}(\widetilde{E})$ is also a Caccioppoli set and $f_{\widetilde{E}}$ is a function of bounded variation.

ii) $\mathrm{Per}(\mathcal{S}(\widetilde{E}); \Omega \times \mathbb{R}) \leq \mathrm{Per}(\widetilde{E}; \Omega \times \mathbb{R})$, with equality if and only if $\mathcal{S}(\widetilde{E}) = \widetilde{E}$, almost everywhere.

Additionally, we will also need a refined statement about the signed distance to graphs.

Lemma 1.3.5. *Suppose $u : \overline{\Omega} \to \mathbb{R}$ is continuous. Then for every $x_0 \in \Omega$ the function*

$$\mathbb{R} \ni z \mapsto \text{sdist}_u(x_0, z) \in \mathbb{R},$$

is strictly increasing.

Proof. Fix $x_0 \in \Omega$, $z_1 < z_2 \in \mathbb{R}$. Suppose for the moment that $z_1 > u(x_0)$ and choose $\bar{x} \in \overline{\Omega}$ such that

$$|(\bar{x}, u(\bar{x})) - (x_0, z_2)| = \text{sdist}_u(x_0, z_2) =: d_2.$$

Either we have $u(\bar{x}) \leq z_1$ and thus $0 < z_1 - u(\bar{x}) < z_2 - u(\bar{x})$ so that

$$\text{sdist}_u(x_0, z_1) \leq |(\bar{x}, u(\bar{x})) - (x_0, z_1)| < |(\bar{x}, u(\bar{x})) - (x_0, z_2)| = d_2.$$

Or otherwise, if $u(\bar{x}) > z_1$, by convexity of Ω and the intermediate valued theorem we find $\tilde{x} \in]x_0; \bar{x}[$ such that $u(\tilde{x}) = z_1$ and thus

$$\operatorname{sdist}_u(x_0, z_1) \leq |x_0 - \tilde{x}| < |x_0 - \bar{x}| \leq d_2.$$

The cases where either $z_1 = u(x_0)$ or $z_2 = u(x_0)$ can easily be discussed by considering the sign of sdist_u at (x_0, z_1) and (x_0, z_2). Finally, the case $z_1 < z_2 < u(x_0)$ can be treated analogous to the first one. □

As a simple corollary of those two lemmas we get:

Proposition 1.3.6. *Suppose $\widetilde{E} \in \widetilde{\mathcal{C}}$ and $h > 0$. Then we have*

i) $\mathcal{S}(\widetilde{E}) \in \widetilde{\mathcal{C}}$.

ii) $\widetilde{\mathcal{F}}_{E_0,h}(\mathcal{S}(\widetilde{E})) \leq \widetilde{\mathcal{F}}_{E_0,h}(E)$ *with equality only if* $\mathcal{S}(\widetilde{E}) = \widetilde{E}$ *(almost everywhere).*

Proof. *i*) Since $\widetilde{E} \in \widetilde{\mathcal{C}}$ we know that on Ω one has $f_{\widetilde{E}} \geq \psi$ and on $\widetilde{\Omega} \setminus \overline{\Omega}$ we have $f_{\widetilde{E}} = \widetilde{u_0}$. Thus, $\mathcal{S}(\widetilde{E}) \in \widetilde{\mathcal{C}}$ as we know by the Lemma 1.3.4 *i*) that $\mathcal{S}(\widetilde{E})$ is again a Caccioppoli set.

ii) Also by Lemma 1.3.4 we know that unless $\widetilde{E}$ is already a subgraph, the perimeter (in $\Omega \times \mathbb{R}$) will be decreased by $\mathcal{S}(\cdot)$. Hence, it remains to show that

$$\int_{\mathcal{S}(\widetilde{E}) \cap (\Omega \times]0,+\infty[)} \operatorname{sdist}_{u_0}(x)\, \mathrm{d}x \leq \int_{\widetilde{E} \cap (\Omega \times]0,+\infty[)} \operatorname{sdist}_{u_0}(x)\, \mathrm{d}x.$$

By Fubini's theorem it suffices to argue slice-wise, more precisely, we need to show that for almost every $y \in \widetilde{\Omega}$:

$$\int_{\mathbb{R}} \operatorname{sdist}_{u_0}(y,z) \chi_{\mathcal{S}(\widetilde{E}) \cap (\{y\} \times \mathbb{R})}(y,z)\, \mathrm{d}z \leq \int_{\mathbb{R}} \operatorname{sdist}_{u_0}(y,z) \chi_{\widetilde{E} \cap (\{y\} \times \mathbb{R})}(y,z)\, \mathrm{d}z.$$

For fixed $y \in \widetilde{\Omega}$ we define $g_y(z) := \mathcal{H}^1\left(\widetilde{E} \cap (\{y\} \times [0,z])\right)$. This allows us to rewrite the left hand side as

$$\int_{\mathbb{R}} \operatorname{sdist}_{u_0}(y,z) \chi_{\mathcal{S}(\widetilde{E}) \cap (\{y\} \times \mathbb{R})}(y,z)\, \mathrm{d}z = \int_{\mathbb{R}} \operatorname{sdist}_{u_0}(y, g_y(z)) \chi_{\widetilde{E} \cap \{y\} \times \mathbb{R}}(y,z)\, \mathrm{d}z.$$

Since for every $y \in \widetilde{\Omega}$ one knows that $g_y(z) \leq \mathcal{H}^1([0,z]) = z$, the remaining inequality follows from the fact that $z \mapsto \operatorname{sdist}_{u_0}(y,z)$ is (strictly) increasing, see Lemma 1.3.5. □

We are now able to prove the following result about minimizers of $\mathcal{F}_{E_0,h}$.

Proposition 1.3.7. *Let $E \in \mathcal{C}$ be a minimizer of $\mathcal{F}_{E_0,h}$. Then E is the subgraph of some $f \in \mathrm{BV}(\Omega)$ with the properties that $f \geq \psi$ a.e. in Ω and $\mathrm{Tr}(f) = u_{0|\partial\Omega}$. Moreover we have*

$$\|f\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}.$$

Proof. By Proposition 1.3.2 we know that there exist minimizers. Proposition 1.3.3, Proposition 1.3.6 and Remark 1.3.1 (together with an approximation argument) give the desired properties of minimizers. □

1.4. Non-Parametric Formulation, Uniqueness and Regularity of Minimizers

In the previous section we learned that the functional $\mathcal{F}_{E_0,h}$ attains its infimum in the set $\mathcal{C}$ and all minimizers are actually subgraphs of BV-functions which attain the boundary data. It is therefore natural to formulate and consider the discretization directly in a non-parametric way. Indeed, the previous results will guarantee the existence of solutions to an associated minimization problems among functions of bounded variation. We recall, that for $f \in \mathrm{BV}(\Omega)$ one defines the (relaxed) area integral as follows:

$$\int_\Omega \sqrt{1+|Df|^2} := \sup\left\{ \int_\Omega \phi_{n+1} + f\mathrm{div}(\phi')\mathrm{d}x \;\middle|\; \phi = (\phi', \phi_{n+1}) \in C_c^1(\Omega;\mathbb{R}^n \times \mathbb{R}) \ , \ \|\phi\|_{L^\infty(\Omega)} \le 1 \right\}.$$

With $u_0, \psi, h > 0$ and $\Omega \subset \mathbb{R}^n$ as introduced in the previous section, we now consider the problem of finding f_1 such that

$$\begin{cases} f_1 \in \mathcal{A}_{\mathrm{BV}} := \{ f \in \mathrm{BV}(\Omega) \ : \ \mathrm{Tr}(f) = u_{0|\partial\Omega}, \, f \ge \psi \text{ a.e.}\}, \\ E(f_1) \le E(f) \quad \forall f \in \mathcal{A}_{\mathrm{BV}}, \end{cases} \tag{1.6}$$

where

$$E(f) := \int_\Omega \sqrt{1+|Df|^2} + \frac{1}{h}\int_\Omega \int_0^{f(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z\,\mathrm{d}x.$$

We can now make our initial claim more precise.

Proposition 1.4.1. *Let $\Omega \subset \mathbb{R}^n$ be open, smooth, bounded and convex, $u_0 \in C^2(\overline{\Omega})$, $\psi \in C^{1,1}(\Omega)$ with $\psi \le u_0$ and $\psi < u_0$ on $\partial\Omega$. Then there exists $f_1 \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ solving problem* (1.6).

Proof. To see this, it suffices to note that every $f \in \mathcal{A}_{\mathrm{BV}}$ gives rise to a Caccioppoli set in $\mathcal{C}$ – namely, via its subgraph – and that the perimeter of this set in $\Omega \times \mathbb{R}$ is equal to the area integral of f, see for instance Theorem 14.6 in [72]. Consequently, by Remark 1.3.2 we have for every $f \in \mathcal{A}_{\mathrm{BV}}$

$$E(f) = \mathcal{F}_{E_0,h}(\mathrm{subgraph}(f)) + \mathit{const}. \tag{1.7}$$

Due to Proposition 1.3.2 and Proposition 1.3.7 we know that there exists some $f_1 \in \mathcal{A}_{\mathrm{BV}}$ such that its subgraph minimizes $\mathcal{F}_{E_0,h}$ among all sets in $\mathcal{C}$, thus in particular among all subgraphs of functions in $\mathcal{A}_{\mathrm{BV}}$. Using (1.7), this translates into $E(f_1) \le E(f)$ for every $f \in \mathcal{A}_{\mathrm{BV}}$. □

As the following example shows, unlike the area integral for functions in $W^{1,1}(\Omega)$, its relaxed version is no longer strictly convex.

Example 1.4.1. Consider $u \equiv 0$, $v = \chi_{[1,2]} \in \mathrm{BV}(]0,3[)$. We have

$$\int_\Omega \sqrt{1+|Du|^2} = 3, \int_\Omega \sqrt{1+|Dv|^2} = 5, \text{ and } \int_\Omega \sqrt{1+|D(1/2(u+v))|^2} = 4,$$

and thus although $\mathrm{Tr}(u) = \mathrm{Tr}(v)$ and $u \neq v$ we get

$$\int_\Omega \sqrt{1+|D(1/2(u+v))|^2} = \frac{1}{2}\left(\int_\Omega \sqrt{1+|Du|^2} + \int_\Omega \sqrt{1+|Dv|^2}\right).$$

Nevertheless, we can exploit the properties of sdist_{u_0} to deduce strict convexity of E.

Lemma 1.4.2. *Suppose $u, v \in \mathcal{A}_{\mathrm{BV}}$ with $u \neq v$, i.e. $|\{u \neq v\}| > 0$ and such that $E(u)$ and $E(v)$ are finite. Then we have*

$$E((1-\lambda)u + \lambda v) < (1-\lambda)E(u) + \lambda E(v) \quad \forall \lambda \in]0,1[.$$

Proof. For u, v as above, $\lambda \in]0,1[$ we set $w := (1-\lambda)u + \lambda v$ and consider some test $\phi = (\phi', \phi_{n+1}) \in C_c^1(\Omega; \mathbb{R}^n \times \mathbb{R})$ with $|\phi(x)| \le 1$ for every $x \in \Omega$. We get

$$\begin{aligned}\int_\Omega \phi_{n+1} + w \operatorname{div}(\phi')\mathrm{d}x &= (1-\lambda)\int_\Omega \phi_{n+1} + u \operatorname{div}(\phi')\mathrm{d}x + \lambda \int_\Omega \phi_{n+1} + v \operatorname{div}(\phi')\mathrm{d}x \\ &\le (1-\lambda)\int_\Omega \sqrt{1+|Du|^2} + \lambda \int_\Omega \sqrt{1+|Dv|^2}.\end{aligned}$$

Taking the supremum over all admissible ϕ we obtain

$$\int_\Omega \sqrt{1+|Dw|^2} \le (1-\lambda)\int_\Omega \sqrt{1+|Du|^2} + \lambda \int_\Omega \sqrt{1+|Dv|^2}. \tag{1.8}$$

Concerning the volume term, we observe that by Lemma 1.3.5 the function $\mathrm{sdist}_{u_0}(x, \cdot)$ is strictly increasing for every $x \in \Omega$. Therefore, for every $x \in \Omega$ the function

$$\mathbb{R} \ni \zeta \mapsto \int_0^\zeta \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z \in \mathbb{R},$$

is strictly convex (cf. Lemma B.7). Thus, for every $x \in \Omega$ such that $u(x) \neq v(x)$ we get

$$\int_0^{w(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z < (1-\lambda)\int_0^{u(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z + \lambda \int_0^{v(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z,$$

while for $x \in \Omega$ with $u(x) = v(x)$ we obviously have

$$\int_0^{w(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z = (1-\lambda)\int_0^{u(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z + \lambda \int_0^{v(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z.$$

Since by assumption $|\{u \neq v\}| > 0$, and $E(u)$, $E(v)$ are both finite, we easily deduce

$$\begin{aligned}&\frac{1}{h}\int_\Omega \int_0^{w(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z\,\mathrm{d}x < \\ &\quad \frac{(1-\lambda)}{h}\int_\Omega \int_0^{u(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z\,\mathrm{d}x + \frac{\lambda}{h}\int_\Omega \int_0^{v(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z\,\mathrm{d}x.\end{aligned} \tag{1.9}$$

Combining (1.8) and (1.9) we deduce the desired $E(w) < (1-\lambda)E(u) + \lambda E(v)$. □

As a corollary, we can now deduce uniqueness of solutions to Problem (1.6).

Proposition 1.4.3. *Problem* (1.6) *of minimizing* E *on* $\mathcal{A}_{\mathrm{BV}}$ *has a unique solution.*

Proof. Existence was already shown in Proposition 1.4.1. Let therefore $u, v \in \mathcal{A}_{\mathrm{BV}}$ be two minimizers of E and set $w := \frac{1}{2}(u+v)$. Observe that $w \in \mathcal{A}_{\mathrm{BV}}$. Unless $u = v$, by strict convexity of E we deduce $E(w) < \frac{1}{2}E(u) + \frac{1}{2}E(v) = E(u)$ which is contradicting the fact that u minimizes E in $\mathcal{A}_{\mathrm{BV}}$. □

Finally, we address the question of regularity for minimizer. At this point, in order to iterate the scheme in the graphical setting we need to make sure that Lemma 1.3.5 can also be applied to sdist_{f_1}, i.e. we should verify that f_1 is at least uniformly continuous. The literature on prescribed curvature problems (with obstacle) offers such results using a priori estimates, consider for instance Theorem 2 in [63].

Proposition 1.4.4. *The unique minimizer of* (1.6) *is Lipschitz continuous.*

However, for the sake of being self-contained we also offer another way to prove this proposition. Let us remark that one could directly study the problem of minimizing E among all functions in

$$\mathcal{A}_{\mathrm{Lip}} := \{u \in \mathrm{Lip}(\Omega) \ : \ u_{|\partial\Omega} = u_{0|\partial\Omega}, \ u \geq \psi \text{ in } \Omega\}.$$

In fact, this problem will be studied systematically in the subsequent chapter and we will prove existence of minimizers in this class (see Theorem 2.2.8). If we can additionally show that the minimal energy in $\mathcal{A}_{\mathrm{Lip}}$ is not bigger than the energy of the minimizer in $\mathcal{A}_{\mathrm{BV}}$, then this provides an alternative proof of Proposition 1.4.4.

Lemma 1.4.5. *Let* f *be the unique minimizer of* E *in* $\mathcal{A}_{\mathrm{BV}}$*. Then we have*

$$E(f) = \inf_{u \in \mathcal{A}_{\mathrm{Lip}}} E(u).$$

Before proving this lemma, let us show why it implies Proposition 1.4.4 if we assume the existence of a minimizer of E in $\mathcal{A}_{\mathrm{Lip}}$.

Alternative proof of Proposition 1.4.4 (assuming Theorem 2.2.8). Let f be the unique minimizer of E in $\mathcal{A}_{\mathrm{BV}}$ and let $u \in \mathcal{A}_{\mathrm{Lip}}$ be the minimizer among all Lipschitz competitors (whose existence will be proved independently in Theorem 2.2.8). By the previous lemma we know that $E(f) = E(u)$. It is straightforward to check that $v := \frac{1}{2}(u+f)$ belongs to $\mathcal{A}_{\mathrm{BV}}$ so that by Lemma 1.4.2, unless $u = f$, we get $E(v) < E(f)$ which would contradict Proposition 1.4.3. □

Let us conclude this chapter with the technical proof of the auxiliary lemma.

Proof of Lemma 1.4.5. Since $\mathcal{A}_{\mathrm{Lip}} \subset \mathcal{A}_{\mathrm{BV}}$ we obviously have $E(f) \leq \inf_{u \in \mathcal{A}_{\mathrm{Lip}}} E(u)$. For the reverse inequality we will construct a sequence $(u_k)_{k \in \mathbb{N}}$ in $\mathcal{A}_{\mathrm{Lip}}$ with the property that

$$\lim_{k \to \infty} E(u_k) = E(f).$$

For $\varepsilon > 0$ let ρ_ε be the standard mollifier and let $f_\varepsilon := \rho_\varepsilon * (f - \psi) + \psi : \Omega_\varepsilon \to \mathbb{R}$. We recall that Note that since $(f - \psi) \geq 0$ also $\rho_\varepsilon * (f - \psi) \geq 0$ and hence $f_\varepsilon \geq \psi$ in Ω_ε. We then choose $\delta_0 > 0$ such that $\Omega_{2\delta} := \{x \in \Omega : \mathrm{dist}_{\partial\Omega}(x) > 2\delta\} \neq \emptyset$ and define for $\delta_0 \geq \delta \geq \varepsilon > 0$

$$v_{\varepsilon,\delta}(x) := \begin{cases} u_0(x), & x \in \Omega \setminus \Omega_\delta, \\ (1 - \frac{d(x)}{\delta})u_0(x) + \frac{d(x)}{\delta} f_\varepsilon(x), & x \in \Omega_\delta \setminus \Omega_{2\delta}, \\ f_\varepsilon(x), & x \in \Omega_{2\delta}. \end{cases}$$

Here we used $d(x) := \mathrm{dist}_{\partial\Omega_\delta}(x)$. In order to rule out a possible ambiguity, we remark that by $\partial\Omega_\delta$ we always denote the set $\partial(\Omega_\delta)$. As $(f - \psi) \in L^1(\Omega)$ we get that $f_\varepsilon \in \mathrm{Lip}(\Omega_\varepsilon)$. Consequently, by using Lemma 1.2.4 twice, we deduce that $v_{\varepsilon,\delta} \in \mathrm{Lip}(\Omega)$. Moreover, by construction $v_{\varepsilon,\delta} = u_0$ on $\partial\Omega$. Since $u_0, f_\varepsilon \geq \psi$ in Ω_δ we get $v_{\varepsilon,\delta} \geq \psi$ in Ω and thus $v_{\varepsilon,\delta} \in \mathcal{A}_{\mathrm{Lip}}$. We set $u_k := v_{\varepsilon_k,\delta_k}$ where $\delta_k = \frac{\delta_0}{k}$ and $\varepsilon_k \leq \delta_k$ to be chosen more specifically at the end of the proof. Let us now observe first of all, that since $u_0 \in C^2(\overline{\Omega})$ and $\delta_k \to 0$ as $k \to \infty$ we get

$$\int_{\Omega\setminus\Omega_{\delta_k}} \sqrt{1 + |\nabla u_0|^2}\mathrm{d}x + \frac{1}{h}\int_{\Omega\setminus\Omega_{\delta_k}} \int_0^{u_0(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z\,\mathrm{d}x \to 0 \quad (k \to \infty). \tag{1.10}$$

Let us next consider the energy of u_k in the region $\Omega_{2\delta_k}$. Extending f_{ε_k} outside of Ω_{ε_k} by zero and denoting this extension also by f_{ε_k} we get that

$$f_{\varepsilon_k} \to f \quad \text{in } L^1(\Omega) \quad (k \to \infty).$$

Therefore

$$\left| \int_{\Omega_{2\delta_k}} \int_0^{f_{\varepsilon_k}(x)} \mathrm{sdist}_{u_0}(x,z)\mathrm{d}z\,\mathrm{d}x - \int_\Omega \int_0^{f(x)} \mathrm{sdist}_{u_0}(x,z)\mathrm{d}z\,\mathrm{d}x \right| =$$
$$\left| \int_\Omega \int_{f(x)}^{f_{\varepsilon_k}(x)} \mathrm{sdist}_{u_0}(x,z)\mathrm{d}x\,\mathrm{d}z \right| \overset{(*)}{\leq} C \int_\Omega |f_{\varepsilon_k} - f|\mathrm{d}x \to 0 \quad (k \to \infty), \tag{1.11}$$

where we used the fact that $\|f\|_{L^\infty(\Omega)} < +\infty$ and hence $\|f_{\varepsilon_k}\|_{L^\infty(\Omega)} \leq C < +\infty$, where C does not depend on k, which allowed us to bound the signed distance in $(*)$. What concerns the area-term, we note that by lower-semicontinuity we get

$$\int_\Omega \sqrt{1 + |Df|^2} \leq \liminf_{k\to\infty} \int_{\Omega_{2\delta_k}} \sqrt{1 + |\nabla f_{\varepsilon_k}|^2}\,\mathrm{d}x. \tag{1.12}$$

On the other hand, we fix $\eta > 0$ arbitrary and choose $\phi = (\phi', \phi^{n+1}) \in C_c^1(\Omega; \mathbb{R}^n \times \mathbb{R})$ with $\|\phi\|_{L^\infty} \leq 1$ such that

$$\int_{\Omega_{2\delta_k}} \sqrt{1 + |\nabla f_{\varepsilon_k}|^2}\,\mathrm{d}x \leq \int_{\Omega_{2\delta_k}} \phi^{n+1} + f_{\varepsilon_k}\mathrm{div}(\phi')\mathrm{d}x + \eta. \tag{1.13}$$

We have

$$\int_{\Omega_{2\delta_k}} \phi^{n+1} + f_{\varepsilon_k}\mathrm{div}(\phi')\mathrm{d}x = \int_{2\Omega_{\delta_k}} \phi_{n+1} + (f-\psi)\mathrm{div}(\rho_{\varepsilon_k}\phi') + \psi\mathrm{div}(\phi')\mathrm{d}x$$
$$= \underbrace{\int_{\Omega_{2\delta_k}} \phi^{n+1}_{\varepsilon_k} + f\mathrm{div}(\phi'_{\varepsilon_k})\mathrm{d}x}_{(I)} + \underbrace{\int_{\Omega_{2\delta_k}} \psi\mathrm{div}(\phi' - \phi'_{\varepsilon_k})\mathrm{d}x}_{(II)} + \underbrace{\int_{\Omega_{2\delta_k}} (\phi^{n+1} - \phi^{n+1}_{\varepsilon_k})\mathrm{d}x}_{(I)},$$

where $\phi'_{\varepsilon_k} := \rho_{\varepsilon_k} * \phi'$ (here we mollify in each component) and $\phi^{n+1}_{\varepsilon_k} := \rho_{\varepsilon_k} * \phi^{n+1}$. Let us discuss the three terms individually. Since ϕ has compact support, the same holds for $\phi_{\varepsilon_k} := (\phi'_{\varepsilon_k}, \phi^{n+1}_{\varepsilon_k})$ for large enough k. Moreover, since $\|\phi\|_{L^\infty} \leq 1$ we also get $\|\phi_{\varepsilon_k}\|_{L^\infty} \leq 1$ so that by definition of the relaxed area integral we get

$$(I) \leq \int_\Omega \sqrt{1+|Df|^2}.$$

For the second and the third term it suffices to note that since $\phi \in C^1_c(\Omega : \mathbb{R}^n \times \mathbb{R})$ we get that ϕ_ε converges to ϕ uniformly as $\varepsilon \to 0$. Therefore we find $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have

$$|(II)| \leq \left\|\phi' - \phi'_{\varepsilon_k}\right\|_{L^\infty(\Omega)} \int_\Omega |\nabla\psi|\,\mathrm{d}x \leq \eta,$$
$$|(III)| \leq \left\|\phi^{n+1} - \phi^{n+1}_{\varepsilon_k}\right\|_{L^\infty(\Omega)} |\Omega| \leq \eta.$$

Combining those three estimates and recalling (1.13) we deduce

$$\limsup_{k\to\infty} \int_{\Omega_{2\delta_k}} \sqrt{1+|\nabla f_{\varepsilon_k}|^2}\,\mathrm{d}x \leq \int_\Omega \sqrt{1+|Df|^2} + 3\eta.$$

Recalling (1.11), (1.12) and the fact that $\eta > 0$ was chosen arbitrarily get that as $k \to \infty$,

$$\lim_{k\to\infty} \int_{\Omega_{2\delta_k}} \sqrt{1+|\nabla f_{\varepsilon_k}|^2}\mathrm{d}x + \frac{1}{h}\int_{\Omega_{2\delta_k}} \int_0^{f_{\varepsilon_k}(x)} \mathrm{sdist}_{u_0}(x,z)\mathrm{d}z\,\mathrm{d}x \to E(f). \tag{1.14}$$

It remains to show that the energy of the interpolation vanishes. Using once more the fact that f_{ε_k}'s are uniformly bounded, similarly to (1.11) we can show that

$$\int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} \int_0^{(1-\frac{d(x)}{\delta_k})u_0(x) + \frac{d(x)}{\delta_k} f_{\varepsilon_k}(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z\,\mathrm{d}x \to 0 \quad (k\to\infty).$$

Using $\frac{d}{\delta_k} \leq 1$ and $|\nabla d| \leq 1$ in $\Omega_{\delta_k} \setminus \Omega_{2\delta_k}$ we can estimate

$$\left|\nabla\left((1-\frac{d}{\delta_k})u_0 + \frac{d}{\delta_k} f_{\varepsilon_k}\right)\right| \leq |\nabla u_0| + |\nabla f_{\varepsilon_k}| + \frac{|f_{\varepsilon_k} - u_0|}{\delta_k},$$

and hence

$$\int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} \sqrt{1+\left|\nabla\left((1-\frac{d}{\delta_k})u_0+\frac{d}{\delta_k}f_{\varepsilon_k}\right)\right|^2}\,\mathrm{d}x$$
$$\leq \int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} 1+|\nabla u_0|+|\nabla f_{\varepsilon_k}|+\frac{|f_{\varepsilon_k}-u_0|}{\delta_k}\,\mathrm{d}x.$$

Since $|\Omega_{\delta_k}\setminus\Omega_{2\delta_k}|\to 0$ for $k\to\infty$, and as $u_0\in C^2(\overline{\Omega})$ we have

$$\int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} 1+|\nabla u_0|\,\mathrm{d}x\to 0\quad (k\to\infty).$$

For any $\phi\in C_c^1(\Omega_{\delta_k}\setminus\overline{\Omega_{2\delta_k}};\mathbb{R}^n)$ with $\|\phi\|_\infty\leq 1$ we note $\phi_{\varepsilon_k}:=\phi*\rho_{\varepsilon_k}$ belongs to $C_c^1(\Omega_{\delta_k-\varepsilon_k}\setminus\overline{\Omega_{2\delta_k+\varepsilon}};\mathbb{R}^n)$ and $\|\phi_{\varepsilon_k}\|_\infty\leq 1$ as well. Therefore, for any such ϕ

$$\int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} f_{\varepsilon_k}\mathrm{div}(\phi)\,\mathrm{d}x=\int_\Omega f\mathrm{div}(\phi_{\varepsilon_k})\,\mathrm{d}x\leq |Df|(\Omega\setminus\overline{\Omega_{2\delta_k+\varepsilon_k}}).$$

We thus derive by taking the supremum among all admissible ϕ that

$$\int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} |\nabla f_{\varepsilon_k}|\,\mathrm{d}x\leq |Df|(\Omega\setminus\overline{\Omega_{2\delta_k+\varepsilon_k}})\to 0\quad (k\to\infty). \tag{1.15}$$

The above convergence follows from the fact that $|Df|$ is a finite Radon measure on Ω and $(\Omega\setminus\overline{\Omega_{2\delta_k+\varepsilon_k}})_{k\in\mathbb{N}}$ is a nested sequence of sets with $\bigcap_{k\in\mathbb{N}}\Omega\setminus\overline{\Omega_{2\delta_k+\varepsilon_k}}=\emptyset$. Let us now discuss the remaining term which we will split as follows:

$$\int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} \frac{|f_{\varepsilon_k}-u_0|}{\delta_k}\,\mathrm{d}x\leq \int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} \frac{|f_{\varepsilon_k}-f|}{\delta_k}\,\mathrm{d}x+\int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} \frac{|f-u_0|}{\delta_k}\,\mathrm{d}x. \tag{1.16}$$

As we know that

$$\int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} |f_\varepsilon-f|\,\mathrm{d}x\to 0\quad (\varepsilon\to 0),$$

for every $k\in\mathbb{N}$, we can choose $\varepsilon_k\leq\delta_k$ small enough such that

$$\int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} |f_{\varepsilon_k}-f|\,\mathrm{d}x\leq \delta_k^2\quad\forall k,$$

which will make sure that the first term on the right hand side of (1.16) vanishes in the limit $k\to\infty$. Concerning the second term, we recall that $\mathrm{Tr}(f)=u_{0|\partial\Omega}$. Therefore, letting $v:=f-u_0$ we note that $v\in\mathrm{BV}(\Omega)$ with trace zero. The convergence of the second term follows immediately once we prove that

$$\int_{\Omega\setminus\Omega_\delta}\frac{|v|}{\delta}\mathrm{d}x\to 0\quad (\delta\to 0). \tag{1.17}$$

We will prove this convergence via an approximation argument. To begin with, let us extend v to all of $\mathbb{R}^n$ by setting

$$\widetilde{v}(x) := \begin{cases} v(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Note that $\widetilde{v}$ belongs to $\mathrm{BV}(\mathbb{R}^n)$ and (recalling that v has trace zero) for any Borel set $V \subset \mathbb{R}^n$ we have $|D\widetilde{v}|(V) = |Dv|(V \cap \Omega)$ (cf. Theorem 5.4.1 in [45]). For $\varepsilon > 0$ we then set $v_\varepsilon := \widetilde{v} * \rho_\varepsilon \in C_c^\infty(\mathbb{R}^n)$. By the co-area formula we get

$$\int_{\Omega \setminus \Omega_\delta} |v_\varepsilon| \mathrm{d}x = \int_0^\delta \int_{\partial \Omega_t} |v_\varepsilon(y)| \, \mathrm{d}\mathcal{H}^{n-1}(y) \, \mathrm{d}t.$$

Without loss of generality, we can assume that δ is so small that $\partial\Omega_\delta = \partial(\Omega_\delta)$ is as smooth as $\partial\Omega$. This allows us to perform the following change of coordinates on the inner integral:

$$\int_{\partial\Omega_t} |v_\varepsilon(y)| \, \mathrm{d}\mathcal{H}^{n-1}(y) = \int_{\varphi(\partial\Omega)} |v_\varepsilon(y)| \, \mathrm{d}\mathcal{H}^{n-1}(y) = \int_{\partial\Omega} |v_\varepsilon(\varphi(z))| J^{\partial\Omega}\varphi(z) \, \mathrm{d}\mathcal{H}^{n-1}(z),$$

where $\varphi := \varphi_t : \partial\Omega \to \partial(\Omega_t)$ is defined as $\varphi(z) := z + t\nu(z)$, with ν denoting the inward pointing unit normal along $\partial\Omega$ and $J^{\partial\Omega}\varphi$ is the tangential Jacobian of φ along $\partial\Omega$ defined as

$$J^{\partial\Omega}\varphi(z) := \sqrt{\det((\nabla^{\partial\Omega}\varphi(z))^T \nabla^{\partial\Omega}\varphi(z))} \quad z \in \partial\Omega.$$

Here $\nabla^{\partial\Omega}$ denotes the tangential gradient. If one smoothly extends φ to some small neighborhood of $\partial\Omega$ (also labeling this extension by φ) we can compute the tangential derivative by using the following relation:

$$\nabla^{\partial\Omega}\varphi = \nabla\varphi - ((\nabla\varphi)\nu) \otimes \nu \quad \text{on } \partial\Omega.$$

This allows us to estimate $J^{\partial\Omega}\varphi(x) \leq \left\|\nabla^{\partial\Omega}\varphi(x)\right\|_{op}^n \leq C(n)\left|\nabla^{\partial\Omega}\varphi(x)\right|^n \leq C|\nabla\varphi(x)|^n$, for $x \in \partial\Omega$. As $\nabla\varphi = Id + t\nabla\nu$ and $\|\nu\|_{L^\infty(\partial\Omega)} \leq C(\Omega)$ we thus get

$$\int_{\Omega\setminus\Omega_\delta} |v_\varepsilon| \mathrm{d}x \leq C(\Omega, n) \int_0^\delta \int_{\partial\Omega} |v_\varepsilon(z + t\nu(z))| \, \mathrm{d}\mathcal{H}^{n-1}(z) \, \mathrm{d}t.$$

Recalling that $\widetilde{v} = 0$ outside of Ω, we see that that $v_\varepsilon(x) = 0$ for every $x \in \mathbb{R}^n$ with $\mathrm{dist}_\Omega(x) \geq \varepsilon$. Hence we can simply bound the integrand in the last expression as follows

$$|v_\varepsilon(x + t\nu(x))| \leq \int_{-\varepsilon}^\delta |(\nabla v_\varepsilon)(x + s\nu(x))| \, \mathrm{d}s \quad \forall t \in [0, \delta].$$

Therefore we get (again by Fubini's theorem) that

$$\begin{aligned} \int_{\Omega\setminus\Omega_\delta} |v_\varepsilon| \mathrm{d}x &\leq C\delta \int_{\partial\Omega} \int_{-\varepsilon}^\delta |(\nabla v_\varepsilon)(x + s\nu(x))| \, \mathrm{d}s \, \mathrm{d}\mathcal{H}^{n-1}(z) \\ &= C\delta \int_{-\varepsilon}^\delta \int_{\partial\Omega} |(\nabla v_\varepsilon)(z + s\nu(z))| \, \mathrm{d}\mathcal{H}^{n-1}(z) \, \mathrm{d}s. \end{aligned}$$

We can employ once more a change of variables and the co-area formula to estimate further and obtain

$$\int_{-\varepsilon}^{\delta}\int_{\partial\Omega} |(\nabla v_\varepsilon)(z+s\nu(z))| \,\mathrm{d}\mathcal{H}^{n-1}(z)\,\mathrm{d}s \le C\int_{\Omega_{-\varepsilon}\setminus\Omega_\delta} |\nabla v_\varepsilon|\,\mathrm{d}x,$$

where we employed the notation $\Omega_{-\varepsilon} = \{x : \mathrm{sdist}_\Omega(x) < \varepsilon\}$ and where the constant C will again depend only on n and Ω. Indeed, the Jacobians that we have to estimate this time are the Jacobian of the maps $\chi : \partial\Omega_t \to \partial\Omega$, $x \mapsto \mathrm{Pr}_{\partial\Omega}(x)$ for $t \in [-\varepsilon, \delta]$ and where $\mathrm{Pr}_{\partial\Omega}$ denotes the (nearest-point-) projection. Noting that those maps are the inverses of the φ's introduced above, the fact that they are smooth follows for instance by the inverse function theorem. Finally taking the derivative on both sides of the identity $\varphi(\chi(x)) = x$ allows us to bound the Jacobian $J^{\partial\Omega_t}\chi(x)$ as we did above for $J^{\partial\Omega}\varphi$. To conclude we obtain

$$\int_{\Omega\setminus\Omega_\delta} |v_\varepsilon|\mathrm{d}x \le C\delta\int_{\Omega_{-\varepsilon}\setminus\Omega_\delta} |\nabla v_\varepsilon|\,\mathrm{d}x \overset{(**)}{\le} C\delta|D\widetilde{v}|(\Omega_{-2\varepsilon}\setminus\Omega_{\delta+\varepsilon}) = C\delta|Dv|(\Omega\setminus\Omega_{\delta+\varepsilon}).$$

Note that in $(**)$ we used the same argument with which we already deduced (1.15). As $\varepsilon \to 0$ we have $v_\varepsilon \to v$ in $L^1(\Omega)$ and thus

$$\int_{\Omega\setminus\Omega_\delta} |v|\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{\Omega\setminus\Omega_\delta} |v_\varepsilon|\,\mathrm{d}x \le C\delta|Dv|(\Omega\setminus\overline{\Omega_\delta}).$$

(1.17) now follows from $|Dv|(\Omega\setminus\overline{\Omega_\delta}) \to 0$ as $(\delta\to 0)$. Thus we showed

$$\int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}} \sqrt{1+|\nabla u_k|^2}\mathrm{d}x + \frac{1}{h}\int_{\Omega_{\delta_k}\setminus\Omega_{2\delta_k}}\int_0^{u_k(x)} \mathrm{sdist}_{u_0}(x,z)\,\mathrm{d}z\,\mathrm{d}x \to 0 \quad (k\to\infty). \tag{1.18}$$

The proposition now follows from (1.10), (1.14) and (1.18) and the definition of u_k. □

Remark 1.4.1. A natural way to extend φ (defined as in the preceding proof) to a neighborhood of $\partial\Omega$ is by setting

$$\varphi(x) := x - \nabla\mathrm{sdist}_{\partial\Omega}(x).$$

In fact, $\mathrm{sdist}_{\partial\Omega}$ is as regular as $\partial\Omega$ (cf. Proposition 1.2.6).

2. The Obstacle Problem for the Prescribed Mean Curvature Equation

The aim of this chapter is to give a self contained account of all the relevant estimates concerning the classical obstacle problem for the minimal surface operator. More precisely, we are going to study the following problem. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex with smooth boundary. We study minimizers of the prescribed mean curvature problem, i.e. minimizers of the energy

$$E(u) := \int_\Omega \left(\sqrt{1 + |\nabla u(x)|^2} + \int_0^{u(x)} H(x,z)\mathrm{d}z \right) \mathrm{d}x, \tag{2.1}$$

in the convex set of competitors

$$\mathcal{A} := \mathcal{A}_{\mathrm{Lip}} := \{u \in \mathrm{Lip}(\Omega) : u_{|\partial\Omega} = g_{|\partial\Omega},\, u \geq \psi \text{ in } \Omega\}, \tag{2.2}$$

where $\psi \in C^{1,1}(\Omega)$ and $g \in C^2(\overline{\Omega})$ are the obstacle and the boundary data respectively which satisfy the compatibility condition $\psi < g$ on $\partial\Omega$ and $H : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded Lipschitz function such that for every $x \in \Omega$:

$$\frac{\partial H}{\partial z}(x,z) \geq 0, \tag{2.3}$$

for $\mathcal{L}^1$-almost every $z \in \mathbb{R}$. As we will see, this condition will guarantee the convexity of E. We would like to point out, that for $H \equiv 0$, the problem of minimizing E among all $u \in \mathcal{A}$ is the one of finding a minimal surface above a given obstacle. This special case is important as we expect it to describe the asymptotic limit of the solution to the parabolic obstacle problem to which we shall come back at the end of Chapter 3.

2.1. Uniqueness of Lipschitz Solutions

Let us assume that u is a minimizer of E in $\mathcal{A}$. Then for every $v \in \mathcal{A}$ and every $t \in [0,1]$ we set $I(t) := E(u + t(v-u))$. As I belongs to $C^1([0,1])$ and achieves its minimum at $t = 0$ we have $I'(0) \geq 0$ which, by a direct computation, leads to a so called *variational inequality.*

$$\int_\Omega \frac{\nabla u \cdot \nabla(v-u)}{\sqrt{1+|\nabla u|^2}} + H(x,u)(v-u)\mathrm{d}x \geq 0 \quad \forall v \in \mathcal{A}. \tag{2.4}$$

This inequality can be viewed as an analog of the usual Euler-Lagrange equation for unconstrained variational problems. As we see next, condition (2.3) implies that solutions to the variational inequality are also minimizers of the energy E.

Proposition 2.1.1. *Let $u \in \mathcal{A}$. Then u is solving* (2.4) *if and only if u minimizes the energy E among all functions in $\mathcal{A}$.*

Proof. The first implication was already shown in the derivation of the variational inequality above. For the reverse, assume that $u \in \mathcal{A}$ solves (2.4). Note, that the map

$$\Omega \times \mathbb{R} \times \mathbb{R}^n \ni (x, z, p) \mapsto F(x, z, p) := \sqrt{1 + |p|^2} + \int_0^z H(x, \zeta)\, \mathrm{d}\zeta$$

is jointly convex in the z and p variable. In other words, for every $x \in \Omega$, the map $F_x(z, p) := F(x, z, p)$, defined on $\mathbb{R} \times \mathbb{R}^n$, is convex. Indeed, the convexity with respect to the p variable is obvious and for the z variable we recall condition (2.3) which is saying that $\partial_z^2 F_x(z, p) \geq 0$ for every $p \in \mathbb{R}^n$ and almost every $z \in \mathbb{R}$. The convexity follows now (cf. Lemma B.8) since the mixed derivatives $\partial_z \partial_{p_i} F_x$ and $\partial_{p_i} \partial_z F_x$ vanish. Consequently, for every $(\zeta, \pi), (z, p) \in \mathbb{R} \times \mathbb{R}^n$ we have

$$F_x(\zeta, \pi) \geq F_x(z, p) + \nabla_{(z,p)} F_x(z, p) \cdot ((\zeta, \pi) - (z, p)),$$

where

$$\nabla_{(z,p)} F_x(z, p) = \left(H(x, z), \frac{p}{\sqrt{1 + |p|^2}} \right).$$

Let now $v \in \mathcal{A}$ and set $(\zeta, \pi) = (v(x), \nabla v(x))$, $(z, p) = (u(x), \nabla u(x))$. We can integrate the above inequality over Ω to obtain

$$E(v) \geq E(u) + \int_\Omega \left(\frac{\nabla u \cdot \nabla (v - u)}{\sqrt{1 + |\nabla u|^2}} + H(x, u)(v - u) \right) \mathrm{d}x.$$

As u solves the variational inequality, the second term on the right hand side is nonnegative and we get the desired minimality of u. □

Remark 2.1.1. For $u \in \mathcal{A}$, we have

$$E(u) = \int_\Omega F(x, u(x), \nabla u(x))\, \mathrm{d}x,$$

where F is defined as in the proof above. Noting that F is strictly convex in p, the above proof can be adapted to derive that the energy E is strictly convex in the sense that whenever $u, v \in \mathcal{A}$ with $u \neq v$ then

$$E(\lambda u + (1 - \lambda) v) < \lambda E(u) + (1 - \lambda) E(v) \quad \forall \lambda \in]0, 1[.$$

Using this remark, we can now give a short proof for the uniqueness of Lipschitz solutions.

Proposition 2.1.2. *There exists at most one minimizer of E in $\mathcal{A}$.*

Proof. Suppose that $u_1, u_2 \in \mathcal{A}$ are two different minimizers of E. By convexity of $\mathcal{A}$ we get that $v := \frac{1}{2}(u_1 + u_2) \in \mathcal{A}$. Using the strict convexity of E (as $u_1 \neq u_2$) we get

$$E(v) < \frac{1}{2}(E(u_1) + E(u_2)) = E(u_1),$$

which is clearly a contradiction to the fact that u_1 minimizes E in $\mathcal{A}$. □

2.2. Existence of Lipschitz Solutions

In this section we prove the existence of Lipschitz continuous solutions to the variational inequality (2.4). We will essentially use the method introduced in [75] by Hartman and Stampacchia for solving elliptic partial differential equations. Those ideas were adapted by Williams in [128] to a class of obstacle problems. Since the obstacle problem considered here is not exactly covered by his assumptions (we need to consider a forcing term that depends explicitly on the space variable x) and for the sake of being self-contained, we will review in detail the strategy as introduced in [75] and [128].

As the space $\mathrm{Lip}(\Omega) \cong W^{1,\infty}(\Omega)$ (and thus $\mathcal{A}$) lacks good compactness properties, we will restrict our problem to a family of smaller sets, look for minimizers of the energy E in these spaces and later show that – under suitable assumptions – these minimizers are also minimizing in $\mathcal{A}$. This is the main idea of the approach proposed by Hartman and Stampacchia in the above mentioned reference.

For $k > 0$ we define $\mathcal{A}_k := \{u \in \mathcal{A} : \mathrm{Lip}(u) \leq k\}$ where $\mathrm{Lip}(u)$ denotes the Lipschitz constant of u. Then we say that $u \in \mathcal{A}_k$ is a minimizer (of E) in $\mathcal{A}_k$ if

$$E(u) \leq E(v) \quad \forall v \in \mathcal{A}_k. \tag{2.5}$$

The advantage of the sets $\mathcal{A}_k$ lies in the fact that they consist of equi-continuous functions which allows us to apply the Arzela-Ascoli theorem to get compactness. Existence of minimizers in $\mathcal{A}_k$ is therefore immediate and the main difficulty is to derive a suitable a priori estimate on the gradient of such solutions which would guarantee the existence of minimizers which lie in the interior of $\mathcal{A}_k$ (in the sense that the Lipschitz constant is strictly smaller than k). As the following proposition shows, this would complete the existence problem for Lipschitz solutions.

Proposition 2.2.1. *Let u be a minimizer in $\mathcal{A}_k$ with $\mathrm{Lip}(u) < k$. Then u is also a minimizer in $\mathcal{A}$.*

Proof. Pick any $v \in \mathcal{A}$ and choose $\varepsilon > 0$ small enough, such that $w := u + \varepsilon(v - u) \in \mathcal{A}_k$. Since we assume that $\mathrm{Lip}(u) < k$ such an ε exists. By minimality of u in $\mathcal{A}_k$ and by convexity of E we get

$$E(u) \leq E(w) \leq (1 - \varepsilon)E(u) + \varepsilon E(v).$$

Rearranging terms yields

$$\varepsilon E(u) \leq \varepsilon E(v),$$

from which the claim follows by positivity of ε. □

For the sake of completeness, we quickly prove that E is lower semicontinuous with respect to L^1-convergence.

Proposition 2.2.2. *Suppose $(u_l)_{l\in\mathbb{N}}$ is a sequence and u a function in $\mathrm{Lip}(\Omega)$ such that $u_l \to u$ in $L^1(\Omega)$ as $(l \to \infty)$. Then we have*

$$E(u) \leq \liminf_{l\to\infty} E(u_l).$$

Proof. We start by considering the area term and by noting that by the dual characterization of the L^1-norm (compare also to the definition of the relaxed area integral for BV-functions in section 1.4) we can write

$$\int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x \\ = \sup\left\{\int_\Omega \phi_0 + \operatorname{div}(\phi')u\,\mathrm{d}x : \phi = (\phi_0,\phi') \in C^1_c(\Omega;\mathbb{R}\times\mathbb{R}^n),\ \|\phi\|_{L^\infty(\Omega)} \le 1\right\}.$$

Let therefore $(\phi_0,\phi') \in C^1_c(\Omega;\mathbb{R}\times\mathbb{R}^n)$ with $\|\phi\|_{L^\infty(\Omega)} \le 1$. Using the convergence of $(u_l)_{l\in\mathbb{N}}$ in $L^1(\Omega)$ we get

$$\int_\Omega \phi_0 + \operatorname{div}(\phi')u\,\mathrm{d}x = \lim_{l\to\infty}\int_\Omega \phi_0 + \operatorname{div}(\phi')u_l\,\mathrm{d}x \le \liminf_{l\to\infty}\int_\Omega \sqrt{1+|\nabla u_l|^2}\,\mathrm{d}x.$$

Taking the supremum among all ϕ as in the dual characterization above yields the lower semi-continuity of the area term. The second term is even easier to handle since it is continuous with respect to L^1-convergence. In fact, to see that $\int_\Omega H(x,u_l) \to \int_\Omega H(x,u)$ as $(l\to\infty)$, it is enough to integrate the inequality

$$|H(x,u_l(x)) - H(x,u(x))| \le \operatorname{Lip}(H)|u(x)-u_l(x)|.$$

□

It is now easily seen, that the notion of minimizers in $\mathcal{A}_k$ is not vacuous if k is sufficiently large.

Proposition 2.2.3. *Let $k \ge \max\{\operatorname{Lip}(g),\operatorname{Lip}(\psi)\}$. Then there exists a unique minimizer of E in $\mathcal{A}_k$.*

Proof. The condition on k guarantees that $\mathcal{A}_k$ is not empty. Indeed, $\max\{g,\psi\}$ belongs to $\mathcal{A}_k$. The remaining part of the proof is an application of the classical *direct method.* Therefore, let us consider a sequence $(u_l)_{l\in\mathbb{N}}$ in $\mathcal{A}_k$ such that $E(u_l) \to \inf_{v\in\mathcal{A}_k} E(v)$ as $(l\to+\infty)$. Since H is bounded from below, also this infimum is bounded from below. Moreover, by definition, the u_l's are equicontinuous. By the simple observation that for any $x_0 \in \partial\Omega$ we have:

$$|u_l(x)| \le |u_l(x_0)| + k|x-x_0| \le \|g\|_{L^\infty(\partial\Omega)} + \operatorname{diam}(\Omega)k,$$

we also get that the u_l's are uniformly bounded. Consequently, by the Arzela-Ascoli theorem, we can extract a subsequence $(u_{l_\nu})_{\nu\in\mathbb{N}}$ converging uniformly to some u. It is straightforward to check that $u \in \mathcal{A}_k$. By the lower-semicontinuity of the energy we get that

$$E(u) \le \liminf_{\nu\to\infty} E(u_{l_\nu}) = \inf_{u\in\mathcal{A}_k} E(u).$$

Therefore, u minimizes E in $\mathcal{A}_k$. To get uniqueness we can argue via the strict convexity of the energy, as we did in the proof of Proposition 2.1.2. □

Remark 2.2.1. The assumption on the boundedness of H was just needed to rule out the possibility that E is unbounded on $\mathcal{A}_k$ (and likewise on $\mathcal{A}$). As already seen in the first chapter, the H we are interested in is (a multiple of) a signed distance function, which, strictly speaking, is unbounded. However, one can easily derive the boundedness of E directly from another property of H which we introduce in (2.13) further down, and we will show how to derive a lower bound on E directly from this property (without assuming boundedness of H).

In the literature on the prescribed mean curvature equation there are a couple of other sufficient conditions on H that guarantee the existence of (weak) solutions. In [71], for instance, the author asks for the existence of two positive constants $\varepsilon_0 < 1$ and z_0 such that for every Caccioppoli set $B \subset \Omega$ one has

$$\int_B H(x, z_0) \geq -(1-\varepsilon_0)\mathrm{Per}(B), \tag{2.6}$$

$$\int_B H(x, -z_0) \leq (1-\varepsilon_0)\mathrm{Per}(B). \tag{2.7}$$

Following section 1.C in [71], let us discuss first, why such a condition is necessary for the existence of smooth solutions to the prescribed mean curvature equation. Suppose $u \in C^2(\overline{\Omega})$ satisfies the associated Euler-Lagrange equation

$$\mathrm{div}\left(\frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}\right) = H(x, u(x)) \quad \forall x \in \Omega.$$

For a fixed Caccioppoli set $B \subset \Omega$ let $(\phi_k)_{k\in\mathbb{N}}$ be a sequence of smooth, compactly supported functions such that $0 \leq \phi_k \leq 1$, $\phi_k \to \chi_B$ almost everywhere as $(k \to \infty)$ and $|\nabla \phi_k|$ converges to $|\nabla \chi_B|$ weakly as measures. (Pick for instance a sequence of regularizations of χ_B as in [97, Proposition 12.20]). Testing the equation with ϕ_k we get

$$-\int_\Omega \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \nabla \phi_k \,\mathrm{d}x = \int_\Omega H(x, u(x))\phi_k \,\mathrm{d}x.$$

Letting $z_0 := \|u\|_{L^\infty(\Omega)}$, $\varepsilon_0 := 1 - \|\nabla u/(1+|\nabla u|^2)^{1/2}\|_{L^\infty(\Omega)}$ and using the monotonicity of H in z, we can therefore estimate

$$-(1-\varepsilon_0)\int_\Omega |\nabla \phi_k| \,\mathrm{d}x \leq \int_\Omega H(x, z_0)\phi_k \,\mathrm{d}x,$$

from where we get (2.6) as we let $k \to \infty$. Similarly we could also derive (2.7). For what concerns the sufficiency of these conditions, we let $u \in \mathcal{A}$ and assume $u \geq 0$. Denote the super-level sets as $U_z = \{x \in \Omega : u(x) > z\}$. Then we have

$$\begin{aligned}
\int_\Omega \int_0^{u(x)} H(x,z)\,\mathrm{d}z\mathrm{d}x &= \int_0^{z_0}\int_{U_z} H(x,z)\mathrm{d}x\mathrm{d}z + \int_{z_0}^{\infty}\int_{U_z} H(x,z)\,\mathrm{d}x\mathrm{d}z \\
&\geq \underbrace{-\int_0^{z_0}\int_\Omega |H(x,z)|\,\mathrm{d}x\mathrm{d}z}_{=:c_0} + \int_{z_0}^{\infty}\int_{U_z} H(x,z)\,\mathrm{d}x\mathrm{d}z \\
&\overset{(2.6)}{\geq} -c_0 - (1-\varepsilon_0)\int_{z_0}^{\infty} \mathrm{Per}(U_z)\mathrm{d}z \\
&\overset{(*)}{\geq} -c_0 - (1-\varepsilon_0)\int_\Omega |\nabla u|\,\mathrm{d}x \\
&\geq -c_0 - (1-\varepsilon_0)\int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x,
\end{aligned}$$

where we used the co-area formula in $(*)$. The general case can be treated similarly by also using (2.7) to control the negative part of u so that we can bound $E(u)$ via

$$E(u) \geq -c_0 + \varepsilon_0 \int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x \quad u \in \mathcal{A}.$$

Some particular hypotheses on H which imply (2.6) and (2.7) are discussed in 1.D to 1.G of [71]. Finally, we note that in the special case that H does only depend on z, things become easier and a necessary and sufficient condition is usually stated similarly to the following one (e.g. (0.6) in [64] or (16.60) in [68]): There exists $\varepsilon_0 > 0$ such that for every Caccioppoli set $B \subset \Omega$ we have

$$\left|\int_B H\,\mathrm{d}x\right| \leq (1-\varepsilon_0)\mathrm{Per}(B).$$

Having established the existence of minimizers in $\mathcal{A}_k$ in Proposition 2.2.3, we will now address the problem of establishing an a priori estimate for such solutions. We will do so by reducing the problem to the simpler task of finding an estimate of the gradient at the boundary via the construction of barriers. However, before we come to the definition of barriers we need to introduce the notion of super- and sub-solutions which then allows us to formulate a useful comparison principle.

Definition 2.2.1. A function $u \in \mathrm{Lip}(\Omega)$ will be called *super-solution* (for the problem of minimizing E) if for all $v \in \mathrm{Lip}(\Omega)$ such that $u_{|\partial\Omega} = v_{|\partial\Omega}$ and $v \geq u$ in Ω we have

$$E(v) \geq E(u).$$

Analogously, u will be called *sub-solution* (for the problem of minimizing E) if for all $v \in \mathrm{Lip}(\Omega)$ such that $u_{|\partial\Omega} = v_{|\partial\Omega}$ and $v \leq u$ in Ω we have

$$E(v) \geq E(u).$$

To simplify the notation, we will henceforth refer to such u's as super- or sub-solution respectively.

Remark 2.2.2. There is an equivalent way of defining super- and sub-solutions which involves the variational inequality rather then the energy. Namely, $u \in \mathrm{Lip}(\Omega)$ is a super-solution (sub-solution respectively) if for all $v \in W_0^{1,\infty}(\Omega)$ with $v \geq 0$ we have:

$$\int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}} + H(x,u)v \,\mathrm{d}x \geq 0 \quad (\leq 0,\ \text{respectively}).$$

The proof of this equivalence is similar to the proof of Proposition 2.1.1

In the following we will often make use of a local version of the energy E. Namely, for any measurable $\Omega' \subset \Omega$ and $u \in \mathcal{A}$ we will use the notation

$$E_{\Omega'}(u) := \int_{\Omega'} \left(\sqrt{1+|\nabla u(x)|^2} + \int_0^{u(x)} H(x,z)\mathrm{d}z \right) \mathrm{d}x. \tag{2.8}$$

Proposition 2.2.4 (Comparison Principle)**.** *Let u be minimizing in $\mathcal{A}_k$ and suppose v is a super-solution with* $\mathrm{Lip}(v) \leq k$*, $\psi \leq v$ in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.*

Similarly, if v is a sub-solution with $\mathrm{Lip}(v) \leq k$ *and $v \leq u$ on $\partial\Omega$, then $v \leq u$ in Ω.*

Remark 2.2.3. Note, that in the previous proposition we allow the sub-solution to penetrate the obstacle.

Proof. To begin with, we suppose v is a super-solution as in the proposition and we set $w_1 := \min\{u, v\}$. Furthermore, we partition the domain Ω into the two sets

$$G := \{x \in \Omega \,:\, u(x) \leq v(x)\} \quad \text{and} \quad B := \{x \in \Omega \,:\, u(x) > v(x)\}.$$

From the minimality of u and since $w_1 \in \mathcal{A}_k$ we easily get that $E(u) \leq E(w_1)$ from which we then derive

$$E_B(u) \leq E_B(v). \tag{2.9}$$

Analogously, we can use the fact that v is a super-solution to get $E(v) \leq E(w_2)$, where $v \leq w_2 := \max\{u, v\}$, from which we then deduce

$$E_B(v) \leq E_B(u). \tag{2.10}$$

Combining (2.9) and (2.10) we get $E_B(u) = E_B(v)$ which can only be true if $|B| = 0$. Indeed, assuming that B has positive measure we could compare u with

$$\widetilde{u}(x) := \begin{cases} u(x), & x \in G, \\ \frac{1}{2}(u(x)+v(x)), & x \in B. \end{cases}$$

Note that $\widetilde{u} \in \mathcal{A}_k$ since we have $\widetilde{u} = \min\{u, \frac{1}{2}(u+v)\}$. Now we can use the strict convexity of E to deduce that $E_B(\widetilde{u}) < E_B(u)$ and hence $E(\widetilde{u}) < E(u)$ which is a contradiction.

The case in which we suppose that v is a sub-solution works very similar. Since this time we want to show that $v \leq u$ we partition the domain Ω into

$$G := \{x \in \Omega \,:\, u(x) \geq v(x)\} \quad \text{and} \quad B := \{x \in \Omega \,:\, u(x) < v(x)\},$$

while we leave the definition of w_1 and w_2 unchanged. From the minimality of u (tested with w_2) we derive $E_B(u) \leq E_B(v)$ and comparing v to $w_1 \leq v$ we the reverse inequality: $E_B(v) \leq E_B(u)$. The contradiction is now obtained if we compare u with $\widetilde{u} := \max\{u, \frac{1}{2}(u+v)\}$. □

In the next step, we show that – under a suitable further assumption on H – a boundary estimate for the gradient of the minimizer extends to a global bound on the Lipschitz constant.

Proposition 2.2.5. *Suppose that either*

$$c_0 := \inf_{x\in\Omega} \operatorname*{ess\,inf}_{z'\in\mathbb{R}} \partial_z H(x,z') > 0, \tag{2.11}$$

or $H \equiv 0$. Moreover, we assume that for u, the minimizer in $\mathcal{A}_k$, we have a boundary gradient estimate in the sense that there exists some $L > 0$ such that:

$$|u(x) - u(y)| \leq L|x-y| \quad \forall x \in \overline{\Omega}\ \forall y \in \partial\Omega.$$

Then we have

$$\operatorname{Lip}(u) \leq \begin{cases} \max\{L, \operatorname{Lip}(\psi), c_0^{-1}\|\nabla_x H\|_{L^\infty(\Omega\times\mathbb{R})}\} & \text{if (2.11) holds,} \\ \max\{L, \operatorname{Lip}(\psi)\} & \text{if } H \equiv 0. \end{cases}$$

Proof. Let $x, y \in \Omega$ and set $h := y - x$, $\tau_h(\Omega) := \Omega + h := \{x + h \,|\, x \in \Omega\}$. We observe that one can shift the whole setting to $\tau_h(\Omega)$, more precisely for $x \in \tau_h(\Omega)$, defining $u_h(x) := u(x-h)$

$$\mathcal{A}^h := \{u \in W^{1,\infty}(\tau_h(\Omega)) \ : \ u = g_h \text{ on } \partial\tau_h(\Omega), \quad u \geq \psi_h \text{ a.e. in } \tau_h(\Omega)\},$$

where g_h and ψ_h are defined analogous to u_h and finally

$$E^h(u) := \int_{\tau_h(\Omega)} \left(\sqrt{1+|\nabla u(x)|^2} + \int_0^{u(x)} H(x-h,z)\mathrm{d}z \right) \mathrm{d}x,$$

for $u \in \mathcal{A}^h$. One can then define in the obvious way also the sets $\mathcal{A}_k^h$ for any $k > 0$. It is straightforward to check that u minimizes E in $\mathcal{A}_k$ if and only if u_h minimizes E^h in $\mathcal{A}_k^h$. Setting $\lambda := |h| \max\{L, \operatorname{Lip}(\psi), c_0^{-1}\|\partial_x H\|_\infty\}$ ($\lambda := |h|\max\{L, \operatorname{Lip}(\psi)\}$ if $H \equiv 0$) we claim that it suffices to prove

$$u \leq u_h + \lambda \quad \text{in } \ \Omega \cap \tau_h(\Omega). \tag{2.12}$$

Indeed, since $y \in \Omega \cap \tau_h(\Omega)$ we could then deduce that that $u(y) - u(x) \leq \lambda$ and the proposition would follow by symmetry in x and y. In order to derive (2.12) we introduce the set $B := \{x \in \Omega \cap \tau_h(\Omega) : u(x) > u_h(x) + \lambda\}$ and we define $w : \Omega \to \mathbb{R}$ as

$$w(x) := \begin{cases} u_h(x) + \lambda & x \in B, \\ u(x) & x \in \Omega \setminus B, \end{cases}$$

and $v : \tau_h(\Omega) \to \mathbb{R}$ as

$$v(x) := \begin{cases} u(x) - \lambda & x \in B, \\ u_h(x) & x \in \tau_h(\Omega) \setminus B. \end{cases}$$

We note that by definition of B and λ, on ∂B we have $u_h + \lambda = u$. To see this, observe that for any $x \in \partial(\Omega \cap \tau_h(\Omega))$ either $x \in \partial\Omega$ or $x - h \in \partial\Omega$. Hence by the boundary estimate we already know that

$$u(x) - u_h(x) = u(x) - u(x-h) \leq L|h|.$$

Thus, w is k-Lipschitz by Lemma 1.2.4. Moreover, on $\partial\Omega$, by the same argument we know that $w = u$ and hence $w = g$. Finally, the choice of λ will make sure that w stays above the obstacle. In fact, this is trivially true on the set $\{w = u\}$. On the other hand, if $w(x) = u_h(x) + \lambda$ we have

$$\begin{aligned} w(x) &= u(x-h) + \lambda \\ &\geq \psi(x-h) + \lambda \\ &\geq \psi(x) \underbrace{-\mathrm{Lip}(\psi)|h| + \lambda}_{\geq 0}. \end{aligned}$$

Therefore $w \in \mathcal{A}_k$. Analogously we can show that $v \in \mathcal{A}_k^h$. By minimality of u we have

$$E(u) \leq E(w).$$

On the other hand, using the minimality of u_h we know that

$$E(w) = E^h(u_h) + (E(w) - E^h(u_h)) \leq E^h(v) + (E(w) - E^h(u_h)).$$

And therefore, using $E^h(v) = E(u) + (E^h(v) - E(u))$, in order to show that $E(w) \leq E(u)$ it suffices to show

$$I := E^h(v) - E(u) + E(w) - E^h(u_h) \leq 0.$$

Using the notation introduced in (2.8), extending it in the obvious way to E^h and noting that outside of B we have $v = u_h$ and $w = u$, we deduce

$$I = E_B^h(v) - E_B^h(u_h) + E_B(w) - E_B(u).$$

Moreover, since on B, $\nabla v = \nabla u$ and $\nabla w = \nabla u_h$ we get

$$I = \int_B \left(\int_0^{v(x)} H(x-h,z)\mathrm{d}z - \int_0^{u_h(x)} H(x-h,z)\mathrm{d}z + \int_0^{w(x)} H(x,z)\mathrm{d}z - \int_0^{u(x)} H(x,z)\mathrm{d}z \right) \mathrm{d}x,$$

which can be written in the more compact form

$$I = \int_B \left(\int_{u_h(x)}^{v(x)} H(x-h,z)\mathrm{d}z - \int_{w(x)}^{u(x)} H(x,z)\mathrm{d}z \right) \mathrm{d}x.$$

Clearly, if $H \equiv 0$, we see that $I = 0$. Otherwise we expand as follows

$$I = \int_B \left(\int_{u_h(x)}^{v(x)} \underbrace{H(x-h,z) - H(x,z)}_{\leq |h| \|\nabla_x H\|_{L^\infty}} \mathrm{d}z + \int_{u_h(x)}^{v(x)} H(x,z)\mathrm{d}z - \int_{w(x)}^{u(x)} H(x,z)\mathrm{d}z \right) \mathrm{d}x.$$

Noting that by a change of variables we get

$$\int_{u_h(x)}^{v(x)} H(x,z)\mathrm{d}z - \int_{w(x)}^{u(x)} H(x,z)\mathrm{d}z = \int_{w(x)}^{u(x)} \underbrace{H(x,z-\lambda) - H(x,z)}_{\leq -\lambda c_0} \mathrm{d}z.$$

Therefore

$$\begin{aligned} I &\leq |h| \|\nabla_x H\|_{L^\infty} \int_B |v(x) - u_h(x)|\mathrm{d}x - \lambda c_0 \int_B |u(x) - w(x)|\mathrm{d}x \\ &= \underbrace{(|h| \|\nabla_x H\|_{L^\infty} - \lambda c_0)}_{\leq 0} \int_B |u - u_h - \lambda|\mathrm{d}x \leq 0. \end{aligned}$$

We thus proved that w is also minimizing E in $\mathcal{A}_k$ but by uniqueness this implies that $w = u$ or in other words $B = \emptyset$ and hence we can conclude (2.12). □

We thus reduced the existence problem to the task of finding suitable a priori bounds for the gradient of solutions at boundary points. In what follows, we will use so called upper and lower *barriers* for our solution to derive such bounds. From now on, we will also impose another condition on H, namely, we assume that there exists a $u_0 \in \mathrm{Lip}(\Omega)$ such that $u_0 = g$ on $\partial\Omega$ and

$$H(x, u_0(x)) = 0 \quad \forall x \in \Omega. \tag{2.13}$$

Remark 2.2.4. First of all, let us come back to Remark 2.2.1. Note, that together with the monotonicity in z, condition (2.13) implies that $H(x,z_1) \geq 0 = H(x,u_0(x)) \geq H(x,z_0)$ for every $z_1 \geq u_0(x) \geq z_0$ and every $x \in \Omega$ so that

$$\int_\Omega \int_0^{u(x)} H(x,z)\mathrm{d}z\mathrm{d}x \geq \int_\Omega \int_0^{u_0(x)} H(x,z)\mathrm{d}z\mathrm{d}x > -\infty.$$

Therefore, E will be bounded from below and we will henceforth always assume this condition. Thus, it is enough to require H to be Lipschitz, but not necessarily bounded (which is exactly what the signed distance, the case we are eventually interested in, satisfies).

Observe, that in the case $H \equiv 0$, every Lipschitz function which coincides with g on $\partial\Omega$ satisfies the above conditions. For simplicity, we will adopt the convention that in this case we take $u_0 = g$. It is easy to see that condition (2.13) yields an L^∞-bound on minimizers in $\mathcal{A}_k$ for sufficiently large k.

Lemma 2.2.6. *Let u be a minimizer of E in $\mathcal{A}_k$ for some $k > 0$. Then we have*

$$\inf_{\Omega} u_0 \leq u(x) \leq \sup_{\Omega} u_0 \quad \forall x \in \Omega. \tag{2.14}$$

Proof. For u as above we consider the set $B := \{x \in \Omega \,:\, u(x) > \sup_\Omega u_0\}$. By continuity of u, B is open and we have $u = \sup_\Omega u_0$ on ∂B. We note that since u is minimizing E, it is also minimizing E_B. Using Remark 2.2.2, (2.3) and (2.13) it is easy to check that $v \equiv \sup_\Omega u_0$ is a supersolution in B and hence by the comparison principle (Proposition 2.2.4) we would get $u \leq v$ in B, which can only be true if $B = \emptyset$. Similarly, one deduces the lower bound by considering the set $\{x \in \Omega \,:\, u(x) < \inf_\Omega u_0\}$ and the subsolution $w \equiv \inf_\Omega u_0$. As already noted in Remark 2.2.3, the fact that w might penetrate the obstacle does not matter. □

Keeping in mind that we henceforth assume (2.13) we can now define upper and lower barriers. We recall the notation $\Omega_t := \{x \in \Omega \,:\, \operatorname{dist}_{\partial\Omega}(x) > t\}$.

Definition 2.2.2. A function $v^+ \in \operatorname{Lip}(\Omega \setminus \Omega_t)$, where $t > 0$ is called upper barrier if

i) $v^+ = g$ on $\partial\Omega$,

ii) $v^+ \geq \sup_\Omega u_0 =: M$ on $\partial\Omega_t$,

iii) v^+ is a super-solution in $\Omega \setminus \Omega_t$,

iv) $v^+ \geq \psi$ in $\Omega \setminus \Omega_t$.

Analogously, we say $v^- \in \operatorname{Lip}(\Omega \setminus \Omega_t)$ is a lower barrier if

i) $v^- = g$ on $\partial\Omega$,

ii) $v^- \leq \inf_\Omega u_0 =: m$ on $\partial\Omega_t$,

iii) v^- is a sub-solution in $\Omega \setminus \Omega_t$.

Note, that we call v a subsolution (resp. supersolution) in $\Omega \setminus \Omega_t$, if v is a subsolution (resp. supersolution) of the problem of minimizing the energy $E_{\Omega\setminus\Omega_t}$.

Once we constructed upper and lower barriers it is straightforward to deduce a bound for the gradient at the boundary.

Proposition 2.2.7. *Suppose that Ω has smooth boundary and that for some $t > 0$ we can construct a super- and a subsolution $v^+, v^- \in \operatorname{Lip}(\Omega \setminus \Omega_t)$ respectively. Then we get a boundary estimate for the gradient of minimizers in $\mathcal{A}_k$ for k large enough. More precisely, whenever u is minimizing in $\mathcal{A}_k$ we get*

$$|u(x) - u(y)| \leq L|x - y| \quad \forall x \in \overline{\Omega}\ \forall y \in \partial\Omega,$$

where $L := \max\{\operatorname{Lip}(v^+), \operatorname{Lip}(v^-)\}$ and $k > L$.

Proof. Let $k > L$ and let u be minimizing in $\mathcal{A}_k$. Note that our assumptions on v^+ and v^-, namely i) and ii), allow us, together with Lemma 2.2.6, to deduce the ordering $v^- \leq u \leq v^+$ on $\partial(\Omega \backslash \Omega_t)$. Using property iii) and the comparison principle (Proposition 2.2.4) we deduce

$$v^-(x) \leq u(x) \leq v^+(x) \quad \forall x \in \Omega \setminus \Omega_t.$$

Since all three functions coincide on the boundary of Ω we get

$$v^-(x) - v^-(y) \leq u(x) - u(y) \leq v^+(x) - v^+(y) \quad \forall x \in \Omega \setminus \Omega_t \quad \forall y \in \partial\Omega.$$

The desired estimate follows immediately once we extend these estimates to the case $x \in \Omega_t$. This can be done by using again Lemma 2.2.6 and the fact that u, v^- and v^+ agree on the boundary. We get

$$m - v^-(y) \leq u(x) - u(y) \leq M - v^+(y) \quad \forall x \in \Omega_t \quad \forall y \in \partial\Omega.$$

Therefore, by property iii), for every $z \in \partial\Omega_t$

$$v^-(z) - v^-(y) \leq u(x) - u(y) \leq v^+(z) - v^+(y) \quad \forall x \in \Omega_t \quad \forall y \in \partial\Omega.$$

The claim follows upon taking $z \in \partial\Omega_t$ with $|z - y| \leq |x - y|$. For t sufficiently small (which can be assumed without loss of generality) such a z exists. □

Finally, we are ready to state the main result of this section.

Theorem 2.2.8. *Let Ω be an open, bounded and convex set with smooth (at least C^2) boundary. Suppose that* (2.11) *holds and that there exists $u_0 \in C^2(\overline{\Omega})$ such that $u_0 = g$ on $\partial\Omega$ and such that* (2.13) *holds. Then there exists a unique minimizer u of E in $\mathcal{A}$. Moreover, we have the following estimate:*

$$\operatorname{Lip}(u) \leq \max\{C, \operatorname{Lip}(\psi), \operatorname{Lip}(u_0), c_0^{-1}\|\nabla_x H\|_{L^\infty(\Omega\times\mathbb{R})}\},$$

for some constant $C = C(\Omega, u_0, \psi)$.

Proof. Uniqueness was already established in Proposition 2.1.2 so it suffices to establish existence here. We will do so by construction suitable barriers. We start by choosing $t_0 > 0$ small enough such that $d(x) := \operatorname{dist}_{\partial\Omega}(x)$ belongs to $C^2(\Omega \setminus \Omega_{t_0})$ and $\Delta d \leq 0$ (cf. Proposition 1.2.6 and appendix B in [72]).

Upper Barrier: Similar to the proof of Theorem 12.10 in [72], we will make the ansatz

$$v^+(x) := u_0(x) + \Psi(d(x)),$$

for some $\Psi \in C^2([0, t_0])$ with $\Psi \geq 0$ and $\Psi(0) = 0$. Obviously, we have $v^+ = u_0 = g$ on $\partial\Omega$. Moreover, since $g > \psi$ on $\partial\Omega$, by possibly decreasing $t_0 > 0$ we can also make sure that $v^+ \geq \psi$ on $\Omega \setminus \Omega_{t_0}$. Next, we observe that v^+ is a supersolution in $\Omega \setminus \Omega_{t_0}$ if we can make sure that

$$\operatorname{div}\left(\frac{\nabla v^+(x)}{\sqrt{1 + |\nabla v^+(x)|^2}}\right) \leq H(x, v^+(x)) \quad \forall x \in \Omega \setminus \Omega_{t_0}. \tag{2.15}$$

Using (2.13), $\Psi \geq 0$ and the monotonicity of H in the second variable, we see that the right hand side is nonnegative. Let us now expand the left hand side of the above equation and find further conditions on Ψ, which will make sure that this term is non-positive. Since

$$\operatorname{div}\left(\frac{\nabla v^+}{\sqrt{1+|\nabla v^+|^2}}\right) = \frac{(1+|\nabla v^+|^2)\Delta v^+ - \langle D^2 v^+ \nabla v^+, \nabla v^+\rangle}{1+|\nabla v^+|^2},$$

it suffices to choose Ψ such that

$$\mathcal{E}(v^+) := (1+\left|\nabla v^+\right|^2)\Delta v^+ - \langle D^2 v^+ \nabla v^+, \nabla v^+\rangle \leq 0.$$

Assuming additionally that $\Psi' \geq 1$ and $\Psi'' > 0$, a lengthy computation shows that for some constant $C = C(u_0, \Omega)$ we get

$$\mathcal{E}(u_0 + \Psi(d)) \leq \Psi''(d) + C(\Psi'(d))^2.$$

Hence, by setting $\Psi(t) := \frac{1}{C}\log(1+\frac{t}{t_0^2})$ we would get the desired $\mathcal{E}(v^+) \leq 0$ as well as $\Psi'' > 0$, $\Psi(0) = 0$ and $\Psi \geq 0$. However, it remains to verify that by (possibly) reducing the value of t_0 further, we can guarantee that $\Psi' \geq 1$ and $v^+ \geq M$ on $\partial\Omega_t$. Concerning the first inequality, it is enough to choose t_0 small enough such that

$$\frac{1}{t_0^2 + t_0} \geq C.$$

In order to have $v^+ \geq M$, on $\partial\Omega_{t_0}$ in view of the definition of v^+, we need impose the condition $\Psi(t_0) \geq M - m$ which will hold whenever

$$\frac{1}{t_0} \geq \exp(C(M-m)) - 1.$$

This implies in particular that it is always possible to construct Ψ with all the desired properties. The previous considerations show that if we choose t_0 small enough, v^+ is an upper barrier. It is straightforward to check that

$$\operatorname{Lip}(v^+) \leq \operatorname{Lip}(u_0) + \operatorname{Lip}(\Psi)|\nabla d| \leq \operatorname{Lip}(u_0) + \frac{1}{Ct_0} \leq C(\Omega, u_0, \psi).$$

Lower Barrier: The construction of the lower barrier can now be done analogously and will be of the form $v^-(x) = u_0(x) - \Psi(d(x))$ where Ψ has to be chosen appropriately. Note, that for the lower barrier it is not relevant to stay above the obstacle, so one finds eventually that

$$\operatorname{Lip}(v^-) \leq C(\Omega, u_0).$$

Let us now pick $k > \max\{L, c_0^{-1}\|\nabla_x H\|_{L^\infty}\}$, where $L := \max\{\operatorname{Lip}(v^+), \operatorname{Lip}(v^-)\}$. By Proposition 2.2.3 there exists a solution of (2.4) in $\mathcal{A}_k$ which we call $\tilde{u}$. Then combining

Proposition 2.2.7, our previous construction of v^+ and v^- and the fact that $k > L$ we deduce a boundary gradient estimate for $\tilde{u}$, more precisely:

$$|\tilde{u}(x) - \tilde{u}(y)| \leq L|x - y| \quad \forall x \in \overline{\Omega} \quad \forall y \in \partial\Omega.$$

Consequently, Proposition 2.2.5 gives us

$$\mathrm{Lip}(\tilde{u}) \leq \max\{L, \mathrm{Lip}(\psi), c_0^{-1}\|\nabla_x H\|_{L^\infty}\} < k,$$

where the strict inequality follows by our choice of k. Finally, Proposition 2.2.1 tells us that $\tilde{u}$ is indeed a solution of (2.4). □

A Remark on the Approach of Kinderlehrer and Stampacchia

The main difficulty in dealing with a variational inequality as in (2.4) arises from the fact that the mean curvature operator

$$u \mapsto -\mathrm{div}\,(a(\nabla u)) \quad \text{with} \quad a(p) := \frac{p}{\sqrt{1 + |p|^2}} \tag{2.16}$$

is not uniformly elliptic (cf. Lemma B.1) and standard existence arguments do not apply. An alternative way to the approach we took in the last section consists in deriving a global *a priori* gradient estimate for smooth solutions which then allows us, roughly speaking, to replace the vector field $a(p)$ by a uniformly elliptic one without affecting the solution to the variational inequality. Using this method, one can prove existence and ($W^{2,p}$-) regularity simultaneously.

In the following, we quickly review this strategy for the case $H = g = 0$ as one can find it, for instance, in the classical monograph [87]. Moreover, we then want to point out, why following this approach does not seem to produce good enough estimates in our case.

In the third chapter of [87] an abstract existence theory for obstacle problems for a class of monotone operators is developed which encompasses quasi-linear operators. However, with the restriction that they need to be uniformly elliptic (or, as they call it, *strongly coercive*). Nevertheless, in the subsequent fourth chapter they also treat the more general case of non-uniformly elliptic quasi-linear operators (in particular the mean curvature operator) although only in the case of zero boundary conditions and without forcing term. Their strategy is the following: Consider the problem of finding a solution to the variational inequality

$$\begin{cases} u \in \mathcal{K}, \\ \displaystyle\int_\Omega a(\nabla u) \cdot \nabla(v - u)\,\mathrm{d}x \geq 0 \quad \forall v \in \mathcal{K}, \end{cases} \tag{2.17}$$

where $\mathcal{K}$ is a suitable class of Lipschitz competitors and $a(p)$ is, for instance, as in (2.16). In a first step (cf. [87, Lemma IV.4.2]) one proves a (global) *a priori* gradient bound for solutions of (2.17) i.e. one shows that there exists some constant C_0 – independent of a – such that any sufficiently smooth solution of the variational inequality satisfies

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C_0.$$

Having such an estimate at our disposal, one can then consider the variational inequality

$$\begin{cases} u \in \mathcal{K}, \\ \displaystyle\int_\Omega \widetilde{a}(\nabla u) \cdot \nabla(v - u)\,\mathrm{d}x \geq 0 \quad \forall v \in \mathcal{K}, \end{cases} \tag{2.18}$$

where $\widetilde{a}$ is any vector field satisfying $\widetilde{a}(p) = a(p)$ for $|p| \leq C_0$ and with linear growth for large p, i.e. in such a way that $\tilde{a}$ gives rise to a uniformly elliptic operator. Now one can apply the theory for uniformly elliptic variational inequalities (cf. [87, Thm. IV.3.6]) to derive existence and regularity of a solution $\tilde{u}$ of (2.18). By applying the a priori bound to $\widetilde{u}$ and recalling the properties of $\widetilde{a}$ we see $a(\nabla\widetilde{u}) = \widetilde{a}(\nabla\widetilde{u})$, i.e. $\widetilde{u}$ is in fact also a solution of (2.17).

Let us now consider a simple example involving a forcing term to see why this strategy causes problems when applied to our case.

Example 2.2.1. Let Ω be the open unit ball in $\mathbb{R}^n$, the obstacle and the boundary data are both given by $g = \psi = 0$, $H(x, z) = -n$ and the vector field is $\hat{a}(p) := \lambda p$, for some $\lambda > 0$. Consider now the variational problem

$$\begin{cases} u \in \mathcal{K} := \{u \in \mathrm{Lip}(\Omega) : u_{|\partial\Omega} = 0,\ u \geq 0 \text{ a.e. in } \Omega\}, \\ \displaystyle\int_\Omega \hat{a}(\nabla u) \cdot \nabla(v - u) + H(x, u)(v - u)\,\mathrm{d}x \geq 0 \quad \forall v \in \mathcal{K}. \end{cases}$$

It is easy to see that the solution is given by $u_\lambda(x) := -\frac{1}{2\lambda}|x|^2 + \frac{1}{2\lambda}$. In fact, $u_\lambda \geq 0$ and it solves

$$-\mathrm{div}(\hat{a}(\nabla u_\lambda(x))) + H(x, u_\lambda(x)) = -\lambda\Delta u - n = 0.$$

Note that $\|\nabla u\|_{L^\infty(\Omega)} = \frac{1}{\lambda}$.

We observe that even if we consider the simplest obstacle and boundary conditions possible, as soon as there is a forcing term involved, any a priori estimate of the gradient will be growing (at least) proportional to $\frac{1}{\lambda}$, where λ is the coercivity constant of $\hat{a}$, i.e. the biggest constant $c > 0$ such that

$$(\hat{a}(p) - \hat{a}(q)) \cdot (p - q) \geq c|p - q|^2 \quad \forall p, q \in \mathbb{R}^n.$$

This is in contrast with the a priori bound derived in [87, Lemma IV.4.2], which is not depending on the involved vector field. The problem with the dependence on the coercivity constant is the following: Suppose you are trying to solve a variational inequality associated to some not uniformly elliptic (i.e. locally coercive) vector field a, then, following the previously outlined strategy, one would start by adapting a outside the set $B_K(0) \subset \mathbb{R}^n$, for a certain level K (coming from the a priori estimate), and construct a uniformly elliptic field $\tilde{a}$ with coercivity constant $\tilde{\nu} \leq \nu(K)$, by which we denote the biggest constant $c > 0$ such that

$$(\tilde{a}(p) - \tilde{a}(q)) \cdot (p - q) \geq c|p - q|^2 \quad \forall p, q \in B_K(0) \subset \mathbb{R}^n.$$

Then one can apply the theory for the uniformly elliptic case to derive existence and regularity of a solution $\tilde{u}$ to the problem associated to $\tilde{a}$. In order to deduce that $\tilde{u}$ solves the actual problem, for the unaltered vector field a, we would need to have

$$\|\nabla \tilde{u}\|_{L^\infty(\Omega)} \leq K.$$

However, as a consequence of the previous example, for a general uniformly vectorfield $\hat{a}$ with coercivity constant $\tilde{\nu}$ we get

$$\|\nabla \tilde{u}\|_{L^\infty(\Omega)} \geq \frac{1}{\tilde{\nu}} \geq \frac{1}{\nu(K)},$$

For the vector field a as in (2.16), according to Lemma B.1 we have

$$\nu(K) = \frac{1}{(1 + K^2)^{\frac{3}{2}}} < \frac{1}{K^3},$$

which therefore implies $\|\nabla \tilde{u}\|_{L^\infty(\Omega)} > K^3$ and hence as soon as $K \geq 1$ the previous strategy of showing that $\tilde{u}$ also solves the obstacle problem associated with a is not directly applicable.

2.3. $W^{2,p}$-Estimate

The Lipschitz regularity for the solutions derived in the previous section will help us since they let our equation become uniformly elliptic. In this section we derive the $W^{2,p}$-estimate for our variational inequality. The following well known result can be found for instance in [113] (see Theorem 7:4.3).

Theorem 2.3.1. *Let $u \in W^{1,\infty}(\Omega)$ be the solution from the previous section. Then for any $p \geq n$, $u \in W^{2,p}(\Omega)$. Moreover, we have*

$$\|u\|_{W^{2,p}(\Omega)} \leq C(\|D^2\psi\|_{L^p(\Omega)} + \|H(\cdot, u)\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} + \|g\|_{W^{2,p}(\Omega)}),$$

where C depends on n, p, $\|\nabla u\|_{L^\infty(\Omega)}$ and Ω.

We will give a detailed proof of this result, following closely the arguments in the above mentioned reference. The following so called *dual estimate* constitutes the main ingredient.

Theorem 2.3.2. *Let $a \in C^1(\Omega \times \mathbb{R}^n; \mathbb{R}^n)$ be uniformly elliptic, i.e. $\exists \nu > 0$ with*

$$(a(x, p) - a(x, q)) \cdot (p - q) \geq \nu|p - q|^2 \quad \forall x \in \Omega \quad \forall\, p, q \in \mathbb{R}^n,$$

and with linear growth (in p), i.e. $\exists K, R > 0$ *such that for some* $f \in L^2(\Omega)$

$$|a(x,p)| \leq K|p| + f(x) \quad \forall x \in \Omega \quad \forall p \in \mathbb{R}^n : |p| \geq R.$$

Furthermore, $\psi \in C^{1,1}(\Omega)$ *with* $\psi < 0$ *on* $\partial\Omega$ *and* $H : \Omega \times \mathbb{R} \to \mathbb{R}$ *as before a Lipschitz function which is increasing in the second variable. We define the operator*

$$A : W_0^{1,2}(\Omega) \ni u \mapsto -\mathrm{div}\,(a(\cdot, \nabla u)) + H(\cdot, u) \in (W_0^{1,2}(\Omega))',$$

and let u solve the variational inequality

$$\begin{cases} u \in \mathcal{K} := \{v \in W_0^{1,2}(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}, \\ \langle Au, v - u \rangle \geq 0 \quad \forall v \in \mathcal{K}, \end{cases} \tag{2.19}$$

where $\langle \cdot, \cdot \rangle$ *is the duality pairing between* $(W_0^{1,2}(\Omega))'$ *and* $W_0^{1,2}(\Omega)$.
Then $Au \in L^2(\Omega)$ *and*

$$0 \leq Au \leq (A\psi)^+ \quad \text{a.e. in } \Omega.$$

Proof. Let us first note that for every $\phi \in W_0^{1,2}(\Omega)$ with $\phi \geq 0$ we have $u + \phi \in \mathcal{K}$. Using this function as a test in the variational inequality we derive

$$\langle Au, \phi \rangle \geq 0. \tag{2.20}$$

Next, we let $\tilde{u}$ be the unique solution of the variational inequality

$$\begin{cases} \tilde{u} \in \tilde{\mathcal{K}} := \{w \in W_0^{1,2}(\Omega) \ : \ w \leq u\}, \\ \langle A\tilde{u} - (A\psi)^+, w - \tilde{u} \rangle \geq 0 \quad \forall w \in \tilde{\mathcal{K}}. \end{cases} \tag{2.21}$$

Existence and uniqueness of $\tilde{u}$ follows essentially from the fact that A defines a coercive operator on the Hilbert space $W_0^{1,2}(\Omega)$ of which $\tilde{\mathcal{K}}$ is a closed, convex and non-empty subset ($\tilde{u} \in \tilde{\mathcal{K}}$). This result goes under the name Lions-Stampacchia theorem. For details, see for instance theorem 4:3.1 in [113]. We will now show that $\tilde{u} = u$. Let us assume for the moment that we additionally know $\tilde{u} \geq \psi$. Then, testing (2.19) with $v = \tilde{u} = u - (u - \tilde{u})$ and testing (2.21) with $w = u = \tilde{u} + (u - \tilde{u})$, we get

$$\begin{aligned} \langle Au, u - \tilde{u} \rangle &\leq 0, \\ \langle (A\psi)^+ - A\tilde{u}, u - \tilde{u} \rangle &\leq 0, \end{aligned}$$

from which, by adding the two inequalities, we deduce

$$\langle Au - A\tilde{u}, u - \tilde{u} \rangle + \underbrace{\langle (A\psi)^+, u - \tilde{u} \rangle}_{\geq 0} \leq 0,$$

which implies – by monotonicity of the operator – that $u - \tilde{u} = 0$. We have to argue additionally that indeed, $\tilde{u} \geq \psi$. To see this, let us test (2.21) with $w = \max\{\psi, \tilde{u}\} = \tilde{u} + (\psi - \tilde{u})^+ \leq u$ to get

$$\langle (A\psi)^+ - A\tilde{u}, (\psi - \tilde{u})^+ \rangle \leq 0.$$

Since $A\psi \le (A\psi)^+$ and by monotonicity of A we deduce that $(\psi - \tilde{u})^+ = 0$ which is equivalent to $\tilde{u} \ge \psi$. Using the just established fact that $\tilde{u} = u$ and testing the variational inequality in (2.21) with $u - \phi$ for any $\phi \in W_0^{1,2}(\Omega)$ with $\phi \ge 0$ we get

$$\langle Au - (A\psi)^+, \phi \rangle \le 0. \tag{2.22}$$

Combining (2.20) and (2.22) we derived that for every $\phi \in W_0^{1,2}(\Omega)$ with $\phi \ge 0$

$$0 \le \langle Au, \phi \rangle \le \langle (A\psi)^+, \phi \rangle. \tag{2.23}$$

Let us now argue why we can pass from this weak form to a point-wise estimate for Au. For arbitrary $\phi \in W_0^{1,2}(\Omega)$ we decompose $\phi = \phi^+ - \phi^-$ to obtain

$$\langle Au, \phi \rangle = \langle Au, \phi^+ \rangle - \langle Au, \phi^- \rangle \le \langle Au, \phi^+ \rangle \le \langle (A\psi)^+, \phi^+ \rangle \le \left\| (A\psi)^+ \right\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)},$$

where we used the fact that $(A\psi)^+ \in L^2(\Omega)$. Similarly, we deduce

$$\langle Au, \phi \rangle \ge -\langle Au, \phi^- \rangle \ge -\left\| (A\psi)^+ \right\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)}.$$

Hence we showed that

$$|\langle Au, \phi \rangle| \le \left\| (A\psi)^+ \right\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)} \quad \forall \phi \in W_0^{1,2}(\Omega).$$

By density of $W_0^{1,2}(\Omega)$ in $L^2(\Omega)$ we can thus extend Au to a bounded linear functional on $L^2(\Omega)$ satisfying the above inequality for every $\phi \in L^2(\Omega)$. By a slight abuse of notation we continue to call this extension Au. Therefore, $Au \in (L^2(\Omega))' \cong L^2(\Omega)$. The missing point-wise inequality now follows from (2.23). Indeed, the fact that $Au \in L^2(\Omega)$ allows us now to localize in the weak inequality. □

In order to apply the previous result to our problem, we need to modify our elliptic vector field in a controlled way to obtain a uniformly elliptic one. For a proof of this technical result, we refer the reader to Lemma III.4.3 in [87].

Lemma 2.3.3. *Let $a \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ be an elliptic vector field i.e.$\forall r > 0\ \exists \nu = \nu(r) \ge 0$ with*

$$(a(p) - a(q)) \cdot (p - q) \ge \nu |p - q|^2 \quad \forall\, p, q \in B_r(0),$$

and let $L > 0$ be any number. Then one can always find a uniformly elliptic vector field $\tilde{a} \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ with the following properties:

i) $a(p) = \tilde{a}(p) \quad \forall |p| \le L.$

ii) $|\tilde{a}(p)| \le C|p| \quad \forall |p| \ge 3L.$

Proof of Theorem 2.3.1. First of all, we choose $\tilde{a}$ according to Lemma 2.3.3 for

$$a(p) = (1 + |p|^2)^{-\frac{1}{2}} p \quad \text{and} \quad L = \|\nabla u\|_{L^\infty(\Omega)}.$$

Let us then define the following translations: $\tilde{u} = u - g$ and $\tilde{\psi} = \psi - g$ as well as the operator $\tilde{A}v := -\mathrm{div}\,(\tilde{a}(v+g)) + H(\cdot, v+g)$. Then it is easy to see that $\tilde{u}$ solves the following obstacle problem

$$\begin{cases} \tilde{u} \in \tilde{\mathcal{K}} := \{v \in W_0^{1,2}(\Omega), v \geq \tilde{\psi}\}, \\ \langle \tilde{A}\tilde{u}, v - \tilde{u}\rangle \geq 0 \quad \forall v \in \tilde{\mathcal{K}}, \end{cases}$$

where

$$\langle \tilde{A}\tilde{u}, v - \tilde{u}\rangle = \int_\Omega \widetilde{a}(\nabla(\tilde{u}+g)) \cdot \nabla(v - \tilde{u}) + H(x, \tilde{u}+g)(v-\tilde{u})\,\mathrm{d}x.$$

In order to verify that we can apply Theorem 2.3.2 we make the following observations: First of all it is clear that the mapping $(x,p) \mapsto \widetilde{a}(p + \nabla g(x))$ is of class C^1. Moreover, $|\tilde{a}(p+\nabla g(x))| \leq C|p| + C\nabla g(x)$ for every x and every $|p| \geq 3L$. To check the uniform ellipticity, we observe that for some $\nu > 0$ we have

$$\begin{aligned} &(\widetilde{a}(p+\nabla g(x)) - \widetilde{a}(q+\nabla g(x))) \cdot (p-q) \\ &\qquad = (\widetilde{a}(p+\nabla g(x)) - \widetilde{a}(q+\nabla g(x))) \cdot (p+\nabla g(x) - (q+\nabla g(x))) \geq \nu|p-q|^2, \end{aligned}$$

where we used the uniform ellipticity of $\widetilde{a}$ in the last inequality. Finally, we see that $(x,z) \mapsto H(x, z+g(x))$ is Lipschitz and increasing in the z variable. Thus, applying Lemma 2.3.2 we get that $\tilde{A}\tilde{u} \in L^2(\Omega)$ and

$$0 \leq \tilde{A}\tilde{u} \leq (\tilde{A}\tilde{\psi})^+ \quad \text{a.e. in } \Omega.$$

Recalling that $\tilde{A}\tilde{u} = -\mathrm{div}\,(a(\nabla u)) + H(\cdot, u)$ and analogous for $\tilde{A}\tilde{\psi}$ we finally deduce that for $C = C(n)$

$$\begin{aligned} |\mathrm{div}\,(a(\nabla u))| &\leq (-\mathrm{div}\,(a(\nabla\psi)))^+ + (H(x,\psi(x)))^+ + |H(x,u(x))| \\ &\leq C(|D^2\psi| + |H(\cdot,u)|). \end{aligned}$$

As the right hand side is in $L^p(\Omega)$ the desired result now follows from the standard regularity theory of quasilinear elliptic equations (cf. part A of the appendix).

□

Some Consequences of the $W^{2,p}$-Estimate

In this subsection we will investigate the point-wise behavior of the second derivative. For u, ψ as before let us introduce the following two definitions:

$$\mathcal{O} := \{x \in \Omega \ : \ u(x) > \psi(x)\},$$

and the coincidence set

$$\Lambda := \{x \in \Omega \ : \ u(x) = \psi(x)\}.$$

Note, that $\mathcal{O}$ is open and Λ is relatively closed in Ω. Moreover, taking into account that $u > \psi$ on $\partial\Omega$ we see that Λ is a closed subset of Ω.

Proposition 2.3.4. *We have* $u \in C^{2,\alpha}_{loc}(\mathcal{O})$ *for every* $0 < \alpha < 1$. *Furthermore,*

$$\operatorname{div}(a(\nabla u)) = H(x, u(x)) \quad \textit{in } \mathcal{O}, \tag{2.24}$$

$$\operatorname{div}(a(\nabla u)) \leq H(x, u(x)) \quad \textit{a.e. in } \Omega, \tag{2.25}$$

$$\operatorname{div}(a(\nabla u)) = \operatorname{div}(a(\nabla \psi)) \quad \textit{a.e. in } \Lambda, \tag{2.26}$$

where, as before, $a(p) := \frac{p}{\sqrt{1+|p|^2}}$.

Proof. Let us start by taking $\phi \in C_c^\infty(\mathcal{O})$. Then, for $\varepsilon \in \mathbb{R}$ such that $|\varepsilon|$ is small enough, we have $v := u + \varepsilon\phi \in \mathcal{A}$. Hence testing the variational inequality (2.4) with v gives

$$\varepsilon \int_\Omega a(\nabla u) \cdot \nabla \phi + H(x, u)\phi \, \mathrm{d}x \geq 0.$$

Since this has to hold for all ε such that $|\varepsilon|$ is sufficiently small, we get

$$\int_\Omega a(\nabla u) \cdot \nabla \phi + H(x, u)\phi \, \mathrm{d}x = 0,$$

i.e. u is a weak solution of

$$-\operatorname{div}(a(\nabla u)) = H(x, u(x)) \quad \textit{in } \mathcal{O}.$$

The $C^{2,\alpha}$-regularity follows now by the standard theory of quasilinear elliptic equations (cf. Theorems A.3 and A.5) and consequently we get (2.24).

Next we let $\phi \in C_c^\infty(\Omega)$ with $\phi \geq 0$. Then for $\varepsilon > 0$ the function $v := u + \varepsilon\phi$ is again an admissible test and by testing the variational inequality we get

$$\int_\Omega a(\nabla u) \cdot \nabla \phi + H(x, u)\phi \, \mathrm{d}x \geq 0.$$

As $u \in W^{2,p}(\Omega)$ we can integrate by parts to derive

$$\int_\Omega \left(-\operatorname{div}(a(\nabla u)) + H(x, u)\right) \phi \, \mathrm{d}x \geq 0.$$

As this inequality holds for all non-negative tests ϕ, (2.25) follows. Finally, (2.26) simply follows from the fact that the first and second derivatives of u and ψ agree almost everywhere in Λ. $\square$

2.4. $C^{1,1}$-Estimate

In this section we discuss the local boundedness of the second derivatives of the solution to our variational inequality. This result was proved independently by Brezis and Kinderlehrer in [20] and by Gerhardt in [62]. Here we give a detailed account of Gerhardt's proof, carefully taking track of the various constants depending on u and its derivatives.

Let us first state and prove two auxiliary lemmas.

Lemma 2.4.1. *Let $\Omega \subset \mathbb{R}^n$ open and bounded, $\phi \in C^2(\Omega)$, $v \in L^1_{\mathrm{loc}}(\Omega)$. Furthermore, we define, for some $1 \le k \le n$:*

$$\phi_h(x) := \frac{1}{h^2}\left(\phi(x + he_k) + \phi(x - he_k) - 2\phi(x)\right) \quad x \in \Omega_h,$$

where $\Omega_h := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > h\}$. Then we have:

i) $\phi_h \to \partial^2_{x_k}\phi$ $(h \to 0)$ locally uniformly.

ii) $\int_{\Omega_h} v\phi_h \,\mathrm{d}x = \int_{\Omega_h} v_h\phi \,\mathrm{d}x$, if $\phi \in C_c(\Omega)$ and for $h < \operatorname{dist}(\operatorname{spt}(\phi), \partial\Omega)$.

Moreover, if $u \in W^{2,p}(\Omega)$ for some $1 \le p < +\infty$, then for any $\Omega' \subset\subset \Omega$ such that $0 < h < \operatorname{dist}(\Omega', \partial\Omega)$ we have $u_h \in L^p(\Omega')$ (u_h defined analogous to ϕ_h above) and

$$\|u_h\|_{L^p(\Omega')} \le C(p)\left\|\partial_k^2 u\right\|_{L^p(\Omega)}.$$

Proof. *i*) Let ϕ be as in the lemma and fix any $\Omega' \subset\subset \Omega$. Note that $\phi \in C^2(\overline{\Omega'})$. Hence, for any $\varepsilon > 0$ there exists $\delta > 0$ only depending on ε such that $\left|\partial_k^2 u(x) - \partial_k^2 u(y)\right| < \varepsilon$ whenever $x, y \in \Omega'$ with $|x - y| < \delta$. Fix now such a pair of ε and δ and let $h < \delta$. Using the differentiability of u we can rewrite for $x \in \Omega'$:

$$\begin{aligned}
\phi_h(x) &= h^{-2}\left(\phi(x + he_k) - \phi(x) + \phi(x - he_k) - \phi(x)\right) \\
&= h^{-2}\left(\int_0^h \partial_k\phi(x + te_k)\,\mathrm{d}t + \int_0^h \partial_k\phi(x - te_k)\mathrm{d}t\right) \\
&= h^{-2}\left(\int_0^h\int_{-t}^t \partial_k^2\phi(x + se_k)\mathrm{d}s\,\mathrm{d}t\right).
\end{aligned}$$

Hence

$$\left|\phi_h(x) - \partial_k^2\phi(x)\right| \le h^{-2}\int_0^h\int_{-t}^t\left|\partial_k^2\phi(x + se_k) - \partial_k^2\phi(x)\right|\mathrm{d}s\,\mathrm{d}t.$$

Since $h < \delta$ we get $|x + se_k - x| < \delta$ and thus

$$\left|\phi_h(x) - \partial_k^2\phi(x)\right| < h^{-2}\int_0^h\int_{-t}^t \varepsilon\,\mathrm{d}s\,\mathrm{d}t = \varepsilon.$$

Recalling that ε did not depend on x, this proves the locally uniform convergence.

ii) This kind of partial-integration formula follows from a simple change of variables.

Finally let $u \in C^2(\overline{\Omega})$. For $1 \le k \le n$ and $h > 0$ fixed let $\Omega' \subset\subset \Omega$ such that $0 < h < \operatorname{dist}(\Omega', \partial\Omega)$. Then for $x \in \Omega'$ we can write as in the proof of *i*):

$$u_h(x) = h^{-2}\left(\int_0^h\int_{-t}^t \partial_k^2 u(x + se_k)\mathrm{d}s\,\mathrm{d}t\right).$$

Taking the p-th power and applying Hölder twice gives

$$|u_h(x)|^p \le h^{-p-1}\int_0^h\int_{-t}^t\left|\partial_k^2 u(x + se_k)\right|^p\mathrm{d}s(2p)^{p-1}\,\mathrm{d}t,$$

from which we conclude by integrating over Ω' and using that $\Omega' \subset \Omega$:

$$\|u_h\|^p_{L^p(\Omega')} \leq \frac{2^{p+1}}{p}\|\partial_k^2 u\|^p_{L^p(\Omega)}.$$

This proves the claim for smooth functions and for general $u \in W^{2,p}(\Omega)$ we argue by approximation. $\square$

Lemma 2.4.2. *Let A and B be two symmetric $n \times n$ matrices and assume that the eigenvalues of B are nonnegative, those of A strictly positive. Then the following inequality holds:*

$$\|B\| \leq \operatorname{tr}(BA)\|A\|\|A^{-1}\|^2,$$

where $\|\cdot\|$ indicates the operator norm.

Proof. Let $v \in \mathbb{R}^n \setminus \{0\}$ and set $w := A^{-1}v$. Since A is symmetric we obtain

$$\langle Bv, v\rangle = \langle BAw, Aw\rangle = \langle ABAw, w\rangle.$$

Since ABA is also symmetric we can decompose w into a sum of mutually orthogonal vectors $w = \sum_{i=1}^n w_i$ where w_i is an eigenvector of ABA with eigenvalue λ_i. We get

$$\langle ABAw, w\rangle = \sum_i^n \lambda_i |w_i|^2 \leq \sum_{i=1}^n \lambda_i |w|^2 = \operatorname{tr}(ABA)|w|^2.$$

The trace of this triple product can be controlled as follows: Let $(v_i)_{i=1,\dots,n}$ be an orthonormal basis of eigenvectors of A with eigenvalues $a_i > 0$. Since the trace is invariant under changes of the basis, we know that $\operatorname{tr}(AB) = \sum_i \langle ABv_i, v_i\rangle$. Using again the invariance of the trace, we get

$$\operatorname{tr}(ABA) = \sum_{i=1}^n \langle ABAv_i, v_i\rangle = \sum_{i=1}^n a_i \langle ABv_i, v_i\rangle \leq \|A\|\operatorname{tr}(AB).$$

Combining the previous computations we have

$$\langle Bv, v\rangle \leq \operatorname{tr}(AB)\|A\|\|A^{-1}\|^2|v|^2. \tag{2.27}$$

From this last inequality we easily derive the desired inequality. Indeed, since B is symmetric we find a basis of eigenvectors. In particular, there exists $\bar{v} \in \mathbb{R}^n \setminus \{0\}$ such that $B\bar{v} = \|B\|\bar{v}$. Therefore

$$\langle B\bar{v}, \bar{v}\rangle = \|B\||\bar{v}|^2,$$

but by (2.27) we also know that

$$\langle B\bar{v}, \bar{v}\rangle \leq \operatorname{tr}(AB)\|A\|\|A^{-1}\|^2|\bar{v}|^2.$$

Hence $\|B\||\bar{v}|^2 \leq \operatorname{tr}(AB)\|A\|\|A^{-1}\|^2|\bar{v}|^2$ and the desired estimate follows since $\bar{v} \neq 0$. $\square$

Proposition 2.4.3. *Let $u \in \mathcal{A}$ be the solution of the variational inequality* (2.4) *constructed in Theorem 2.2.8. Then $u \in C^{1,1}_{loc}(\Omega)$ and for $\Omega' \subset\subset \Omega$ we have for any $p < \infty$*

$$\left\|D^2 u\right\|_{L^\infty(\Omega')} \leq C \max\left\{\|u\|_{L^\infty(\Omega)} + \left\|D^2 u\right\|^2_{L^{2p}(\Omega)} + \|H(\cdot, u(\cdot))\|_{W^{1,\infty}(\Omega)}, \left\|D^2\psi\right\|_{L^\infty(\Omega)}\right\},$$

where the constant C depends on Ω', n, p and $\mathrm{Lip}(u)$.

Remark 2.4.1. Under the assumptions of Theorem 2.2.8, by the previous section, we already know that u belongs to $W^{2,p}(\Omega)$ for any finite p.

Proof. Let $\Lambda := \{x \in \Omega : u = \psi\}$ and observe that since u and ψ are continuous, Λ is closed. Then choose Ω' open such that

$$\Lambda \subset\subset \Omega' \subset\subset \Omega.$$

Such Ω' exist since by the assumption $\psi < g$ on $\partial\Omega$ we know that Λ has a positive distance from the boundary of Ω. Our aim is to bound the (generalized) second derivatives of u in $L^\infty(\Omega')$. Clearly, on Λ we know that $u = \psi$ and hence in the interior of Λ we get for free that $D^2 u = D^2\psi$. Therefore, we focus now on $\Omega' \setminus \Lambda$.

In order to make the subsequent part of the proof more transparent we will give now a rough summary of how we proceed. In a first step, we show that as a simple consequence of the variational inequality, u_h weakly solves an equation of the form

$$\operatorname{div}(A\nabla u_h + F) = 0.$$

Applying a suitable version of the weak maximum principle, we can bound u_h from below and since we are working in that part of the domain, where u is solving an equation and hence is smooth, this bound translates into a bound on the pure second derivatives of u. To pass from this one-sided bound to a full bound of all second order derivatives is then achieved by an argument involving both Lemma 2.4.2 and the equation solved by u.

Let us start by choosing $\phi \in C^\infty_c(\Omega' \setminus \Lambda)$. Without relabeling, we extend ϕ by 0 to all of $\mathbb{R}^n$. For $0 < h < \operatorname{dist}(\Omega', \partial\Omega)$ set

$$\phi_h(x) := \frac{1}{h^2}\left(\phi(x + he_1) + \phi(x - he_1) - 2\phi(x)\right) \quad \text{for } x \in \overline{\Omega}.$$

Note, that ϕ_h vanishes on $\partial\Omega$ and hence $u + \varepsilon\phi_h = g$ on $\partial\Omega$ for any $\varepsilon \in \mathbb{R}$. Moreover, we have $\operatorname{spt}(\phi_h) \subset \{y \in \Omega : \operatorname{dist}(y, \operatorname{spt}(\phi)) \leq h\}$. Thus, for $h < \operatorname{dist}(\operatorname{spt}(\phi), \Lambda)$ we get $\operatorname{spt}(\phi_h) \subset \Omega \setminus \Lambda$. Noting that $m := \min_{\operatorname{spt}(\phi_h)} u - \psi > 0$, we deduce that for $|\varepsilon| < m^{-1}\|\phi_h\|_{L^\infty}$,

$$u_\varepsilon := u + \varepsilon\phi_h \geq \psi \quad \text{in } \Omega.$$

Altogether, we get that for ε small enough, $u_\varepsilon \in \mathcal{A}$. Testing the variational inequality with u_ε we get

$$\varepsilon \int_\Omega a(\nabla u) \cdot \nabla\phi_h + H(x, u)\phi_h \,\mathrm{d}x \geq 0,$$

where $a(p) := \frac{p}{\sqrt{1+|p|^2}}$, as before. Since this holds for any ε small enough, we deduce

$$\int_\Omega a(\nabla u) \cdot \nabla \phi_h + H(x,u)\phi_h \, \mathrm{d}x = 0.$$

We make the observation that $\nabla \phi_h = (\nabla \phi)_h$ since both operations are linear. Next, we let w be the Newtonian potential of $x \mapsto H(x, u(x))$ i.e. w satisfies

$$-\Delta w(x) = H(x, u(x)) \quad \text{a.e. in } \Omega.$$

By the classical theory of Calderon-Zygmund (cf. section 9.5 in [68]) we have the estimate

$$\left\| D^3 w \right\|_{L^p(\Omega)} \leq C(n,p) \| H(\cdot, u) \|_{W^{1,\infty}(\Omega)}.$$

These two remarks, together with Lemma 2.4.1 *ii*), applied in each component yields

$$\int_\Omega [(a(\nabla u))_h + \nabla(w_h)] \cdot \nabla \phi = 0. \tag{2.28}$$

Let us expand the first term.

$$(a(\nabla u))_h = \frac{1}{h^2} \left[a(\nabla u(x+he_1)) - a(\nabla u(x)) + a(\nabla u(x-he_1)) - a(\nabla u(x)) \right].$$

The first two terms on the right hand side can be dealt with by using the fundamental theorem of calculus and setting $v_h(t,x) := tu(x+he_1) + (1-t)u(x)$ for $(t,x) \in [0,1] \times \Omega'$:

$$\begin{aligned} a(\nabla u(x+he_1)) - a(\nabla u(x)) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} a(t(\nabla u(x+he_1)) + (1-t)\nabla u(x)) \, \mathrm{d}t \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} a(t(\nabla u(x+he_1)) + (1-t)\nabla u(x)) \, \mathrm{d}t \\ &= \int_0^1 Da(\nabla v_h(t,x)) \cdot (\nabla u(x+he_1) - \nabla u(x)) \, \mathrm{d}t. \end{aligned}$$

Adding and subtracting $A_h(x) \cdot [\nabla u(x-he_1) - \nabla u(x)]$, where

$$A_h(x) := \int_0^1 Da(\nabla v_h(t,x)) \, \mathrm{d}t \quad \text{for } x \in \Omega',$$

we get

$$a(\nabla u(x+he_1)) - a(\nabla u(x)) = h^2 A_h(x) \cdot \nabla(u_h) - A_h(x) \cdot [\nabla u(x-he_1) - \nabla u(x)].$$

Treating similarly also the third and fourth term in the expansion of $(a(\nabla u))_h$, we get

$$(a(\nabla u))_h = A_h(x) \cdot \nabla(u_h) + \frac{1}{h^2}(A_{-h}(x) - A_h(x)) \cdot [\nabla u(x-he_1) - \nabla u(x)].$$

Inserting back in (2.28) we see that

$$\int_\Omega [A_h \cdot \nabla(u_h) + F] \cdot \nabla\phi = 0,$$

where $F(x) := \frac{1}{h^2}(A_{-h}(x) - A_h(x)) \cdot [\nabla u(x - he_1) - \nabla u(x)] + \nabla w_h(x)$ for $x \in \Omega'$. We claim that $F \in L^p(\Omega')$ for any $1 < p < +\infty$ (with L^p-norm independent of h) and that A_h defines a symmetric, bounded and uniformly elliptic matrix. Indeed, since A_h is defined as an (integral) average, it suffices to check the properties of Da. Noting that $a = \nabla f$ for $f(p) = (1 + |p|^2)^{1/2}$ we see that $Da = D^2 f$ is symmetric. Moreover, from

$$Da(p) = \frac{1}{(1+|p|^2)^{\frac{2}{3}}}\left((1+|p|^2)\mathrm{id} - p \otimes p\right),$$

one can read of the eigenvalues $(1+|p|^2)^{-\frac{3}{2}}$ (multiplicity 1) and $(1+|p|^2)^{-\frac{1}{2}}$ (multiplicity $n-1$), cf. Lemma B.1. Finally, since

$$|v_h(t,x)| \leq t|\nabla u(x + he_1)| + (1-t)|\nabla u(x)| \leq \|\nabla u\|_{L^\infty(\Omega)},$$

we see that

$$\langle \xi, A_h(x) \cdot \xi \geq \lambda|\xi|^2\rangle \quad \forall \xi \in \mathbb{R}^n \quad \forall x \in \Omega,$$

with $\lambda := (1 + \|\nabla u\|_\infty^2)^{-\frac{3}{2}} > 0$. Let us now discuss the boundedness of F in $L^p(\Omega')$. First of all, we fix some $1 < p < \infty$ and consider the term containing w. Recall that $w \in W^{3,p}(\Omega)$ and consequently $\nabla w \in W^{2,p}(\Omega)$. By the fact that $\nabla(w_h) = (\nabla w)_h$ and the last part of Lemma 2.4.1 (applied to each component) we get the bound

$$\|\nabla w_h\|_{L^p(\Omega')} \leq C\left\|D^2(\nabla w)\right\|_{L^p(\Omega)} \leq C\|H(x, u(x))\|_{W^{1,\infty}(\Omega)}.$$

In the other term we split the prefactor h^{-2} to the two brackets and bound as follows:

$$\begin{aligned}
\frac{1}{h}|A_{-h}(x) - A_h(x)| &= \frac{1}{h}\left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} Da(\nabla v_{-h}(t,x)) - Da(\nabla v_h(t,x))\,\mathrm{d}t\right| \\
&\leq \frac{1}{h}\int_0^1\int_0^1 \left|D^2 a((1-s)\nabla v_h + s\nabla v_{-h})\right|\mathrm{d}s|\nabla v_{-h} - \nabla v_h|\,\mathrm{d}t \\
&\leq \frac{1}{h}\int_0^1 \left\|D^2 a\right\|_{L^\infty(\Omega)} t|\nabla u(x - he_1) - \nabla u(x + he_1)|\mathrm{d}t \\
&\leq \frac{C}{h}|\nabla u(x - he_1) - \nabla u(x + he_1)|.
\end{aligned}$$

By Proposition A.1 *i)* (difference quotients) and since $\nabla u \in W^{1,p}(\Omega; \mathbb{R}^n)$ we can therefore conclude

$$\left\|h^{-1}(A_{-h}(x) - A_h(x))\right\|_{L^p(\Omega')} \leq C\left\|D^2 u\right\|_{L^p(\Omega)}.$$

Analogously, we can now bound the remaining factor, namely

$$\left\|h^{-1}(\nabla u(x - he_1) - \nabla u(x))\right\|_{L^p(\Omega')} \leq C\left\|D^2 u\right\|_{L^p(\Omega)}.$$

We remark that if $f_1, f_2 \in L^{2p}$ for all $p < +\infty$ then their product belongs to L^p for every finite p since by Hölder:

$$\|f_1 f_2\|_{L^p}^p = \int |f_1 f_2|^p \leq \left(|f_1|^{2p}\right)^{\frac{1}{2}} \left(\int |f_2|^{2p}\right)^{\frac{1}{2}} = \|f_1\|_{L^{2p}}^p \|f_2\|_{L^{2p}}^p,$$

which implies

$$\|f_1 f_2\|_{L^p} \leq \|f_1\|_{L^{2p}} \|f_2\|_{L^{2p}}.$$

Combining this fact with the previous estimates we end up with

$$\|F\|_{L^p(\Omega')} \leq C \left(\left\|D^2 u\right\|_{L^{2p}(\Omega)}^2 + \|H(x, u(x))\|_{W^{1,\infty}(\Omega)} \right).$$

Therefore, noting that $u_h \in W^{1,2}(\Omega' \setminus \Lambda) \cap C^0(\overline{\Omega \setminus \Lambda})$, we can apply the maximum principle in the version of Proposition A.7 to deduce that on $\Omega' \setminus \Lambda$

$$u_h \geq \inf_{\partial(\Omega' \setminus \Lambda)} (u_h^-) - C(1 + \|\nabla u\|_{L^\infty(\Omega)}^2)^{\frac{3}{2}} \left(\left\|D^2 u\right\|_{L^{2p}(\Omega)}^2 + \|H(x, u(x))\|_{W^{1,\infty}(\Omega)} \right).$$

In the next step we will show that we can bound $\inf_{\partial(\Omega' \setminus \Lambda)} u_h^-$ from below. Recalling that $\Lambda \subset\subset \Omega'$ we get $\partial(\Omega' \setminus \Lambda) = \partial\Lambda \cup \partial\Omega'$. Suppose first, that $x \in \partial\Lambda$ and hence $u(x) = \psi(x)$. Using $u \geq \psi$ we get

$$\begin{aligned} u_h(x) &= h^{-2} \left(u(x + he_1) + u(x - he_1) - 2\psi(x)\right) \\ &\geq h^{-2} \left(\psi(x + he_1) + \psi(x - he_1) - 2\psi(x)\right). \end{aligned}$$

Assuming for the moment that ψ is in $C^2(\overline{\Omega})$, the mean value theorem tells us that for some $\xi_0 \in [x; x + he_1]$

$$\psi(x + he_1) = \psi(x) + h\partial_1\psi(x) + \frac{1}{2}h^2\partial_1^2\psi(\xi_0),$$

and for some $\xi_1 \in [x - he_1; x]$

$$\psi(x - he_1) = \psi(x) - h\partial_1\psi(x) + \frac{1}{2}h^2\partial_1^2\psi(\xi_1).$$

Adding these equations and rearranging terms gives

$$h^{-2} \left(\psi(x + he_1) + \psi(x - he_1) - 2\psi(x)\right) = \frac{1}{2} \left(\partial_1^2\psi(\xi_0) + \partial_1^2\psi(\xi_1)\right),$$

from which we conclude (for the general case by approximation) that

$$u_h(x) \geq -\left\|D^2\psi\right\|_{L^\infty(\Omega)}.$$

The case $x \in \partial\Omega'$ is even simpler because $\partial\Omega'$ is contained in $\{u > \psi\}$ on which u is (locally) $C^{2,\alpha}$ (cf. Proposition 2.3.4). Thus we know $u_h(x) \geq -\sup_{\partial\Omega'} \left|D^2 u\right|$. Moreover, since $\partial\Omega'$ has positive distance to the boundary of $\Omega \setminus \overline{\{u = \psi\}}$ we can use interior Schauder-estimates to get

$$u_h(x) \geq -C(\|u\|_{L^\infty(\Omega)} + \|H(\cdot,u)\|_{W^{1,\infty}}),$$

where C is a constant depending on n, Ω' and $\mathrm{Lip}(u)$.

Altogether, we end up with: For every $x \in \Omega' \setminus \Lambda$, we have

$$\begin{aligned} u_h(x) \geq & - C\Big(\max\Big\{ \|u\|_{L^\infty(\Omega)} + \|H(\cdot,u)\|_{W^{1,\infty}}), \left\|D^2\psi\right\|_{L^\infty(\Omega)} \Big\} \\ & - \left\|D^2u\right\|^2_{L^{2p}(\Omega)} - \|H(\cdot,u)\|_{W^{1,\infty}(\Omega)} \Big) \\ \geq & - \underbrace{C\left(\max\left\{ \|u\|_{L^\infty} + \left\|D^2u\right\|^2_{L^{2p}} + \|H(\cdot,u)\|_{W^{1,\infty}}), \left\|D^2\psi\right\|_{L^\infty} \right\} \right)}_{C_1}, \end{aligned}$$

where the constant $C > 0$ depends now on n, p, Ω' and $\mathrm{Lip}(u)$. Since u is C^2 on $\Omega' \setminus \Lambda$ we can use Lemma 2.4.1 *i)* and pass into the limit $h \to 0$ to deduce that for all $x \in \Omega' \setminus \Lambda$

$$\partial_1^2 u(x) \geq -C_1.$$

As we could replace e_1 by any unit vector ξ, we get

$$\langle \xi, D^2u(x) \cdot \xi \rangle \geq -C_1 \quad \forall x \in \Omega' \setminus \Lambda \quad \forall \xi \in \mathbb{R}^n : \ |\xi| = 1.$$

Using Lemma 2.4.2, we will now extend this bound also to the mixed derivatives. More precisely, we fix $x_0 \in \Omega' \setminus \Lambda$ and set $B = D^2u(x_0) + C_1 I_n$, $A = D_p a(\nabla u(x_0))$. Clearly, they satisfy the hypotheses of the lemma. Moreover, since the eigenvalues of A are $\left(1 + |\nabla u(x_0)|^2\right)^{-\frac{3}{2}}$ and $\left(1 + |\nabla u(x_0)|^2\right)^{-\frac{1}{2}}$ we get

$$\|A\| \leq \left(1 + |\nabla u(x_0)|^2\right)^{-\frac{1}{2}} \quad \text{and} \quad \left\|A^{-1}\right\| \leq \left(1 + |\nabla u(x_0)|^2\right)^{\frac{3}{2}},$$

and hence

$$\|A\|\|A^{-1}\|^2 \leq \left(1 + |\nabla u(x_0)|^2\right)^{\frac{5}{2}}.$$

It remains to estimate the trace of $BA = D^2u D_p a(\nabla u) + C_1 D_p a$. Note that

$$\mathrm{tr}(BA) = \mathrm{tr}(D^2u D_p a(\nabla u)) + C_1 \mathrm{tr}(D_p a(\nabla u)).$$

The second term can be bounded by using the fact that the trace is invariant under changes of coordinates and the precise knowledge about the multiplicities of the eigenvalues of A. More precisely,

$$\mathrm{tr}(D_p a(\nabla u(x_0))) \leq \left(1 + |\nabla u(x_0)|^2\right)^{-\frac{1}{2}} + (n-1)\left(1 + |\nabla u(x_0)|^2\right)^{-\frac{3}{2}} \leq n.$$

For the first term, we recall that in $\Omega' \setminus \Lambda$ the function u is C^2 so that we can differentiate it twice to get that $\operatorname{div}(a(\nabla u)) = \operatorname{tr}\left(D^2 u D_p a(\nabla u)\right)$. Therefore, we can use the fact that u is solving an equation in $\Omega' \setminus \Lambda$ to get altogether

$$\operatorname{tr}(BA) \leq H(x_0, u(x_0)) + nC_0,$$

from which we derive by Lemma 2.4.2 that

$$\left\|D^2 u(x_0) + C_1 I_n\right\| \leq (H(x_0, u(x_0)) + nC_1)\left(1 + |\nabla u(x_0)|^2\right)^{\frac{5}{2}}.$$

By the fact that $|A_{i,j}| \leq \|A\|$ we finally deduce that for every $x \in \Omega' \setminus \Lambda$ we have for every pair of indices (i, j):

$$|\partial_{ij} u(x_0)| \leq C_1 + (\|H(\cdot, u)\|_\infty + nC_0)\left(1 + |\nabla u(x_0)|^2\right)^{\frac{5}{2}} \leq CC_1.$$

Moreover, since $u = \psi$ in Λ and consequently $D^2 u = D^2 \psi$ a.e. in Λ, we finally get

$$\left\|D^2 u\right\|_{L^\infty}(\Omega') \leq \max\{CC_1, \left\|D^2\psi\right\|_\infty\}.$$

$\square$

Remark 2.4.2. As a consequence of the previous theorem we would also get immediately a global bound on the $C^{1,1}$-norm of u. Indeed, for $r > 0$ small enough, by our assumptions on g and ψ, we know that $\Omega \setminus \Omega_r \subset \mathcal{O}$. Thus, we can apply Hölder-estimates up to the boundary to deduce that we even have $u \in C^{2,\alpha}(\Omega \setminus \Omega_r)$, for every $\alpha < 1$ and r small enough, such that $\Omega \setminus \Omega_r$ is sufficiently regular.

2.5. Minimal Surfaces above an Obstacle

As an easy corollary of the theory we developed in this chapter we can also show the existence of minimal surfaces above obstacles. Letting Ω, g, ψ and $\mathcal{A}$ be as introduced at the beginning of this chapter, we have the following proposition.

Proposition 2.5.1. *There exists a unique $v \in \mathcal{A}$ such that*

$$\int_\Omega \sqrt{1 + |\nabla v|^2}\,\mathrm{d}x \leq \int_\Omega \sqrt{1 + |\nabla w|^2}\,\mathrm{d}x \quad \forall w \in \mathcal{A}.$$

Moreover, $v \in C^{1,1}(\Omega)$.

Proof. Uniqueness was already shown in Proposition 2.1.2. In order to get existence, we just need to adapt the proof of Theorem 2.2.8 to the case $H \equiv 0$. Essentially, we just need to note, that the barriers v^+, v^- we constructed in that proof, work also in this case if we recall the convention $u_0 = g$. All the results used in the rest of the proof, were already formulated to hold also in the case $H \equiv 0$ and the claimed regularity follows from the regularity results in the previous two sections. $\square$

At the end of the next chapter (when we identify the asymptotic limit of the distributional solution of our parabolic obstacle problem), we will need another characterization of minimal surfaces above obstacles.

Lemma 2.5.2. *For $v \in \mathcal{A}$ and with $\Lambda := \{x \in \Omega \,:\, v(x) = \psi(x)\}$ the following are equivalent:*

i) $\displaystyle\int_\Omega \sqrt{1+|\nabla v|^2}\,\mathrm{d}x \le \int_\Omega \sqrt{1+|\nabla w|^2}\,\mathrm{d}x \quad \forall w \in \mathcal{A}.$

ii) $\displaystyle\int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla(w-v)\,\mathrm{d}x \ge 0 \quad \forall w \in \mathcal{A}.$

iii) $\displaystyle\int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla\phi\,\mathrm{d}x \ge 0 \quad \forall \phi \in W_0^{1,\infty}(\Omega):\ \phi \ge 0 \textit{ in } \Lambda.$

iv) $\displaystyle\int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla\phi\,\mathrm{d}x \ge 0 \quad \forall \phi \in C_c^{\infty}(\Omega):\ \phi \ge 0 \textit{ in } \Lambda.$

Proof. The equivalence of *i)* and *ii)* is a special case of Proposition 2.1.1 with $H \equiv 0$. Let us quickly show that *iii)* implies *ii)*. Fix $w \in \mathcal{A}$ and define $\phi := w - v$. Since w and v are both Lipschitz and agree on the boundary, $\phi \in W_0^{1,\infty}(\Omega)$. Moreover, for $x \in \Lambda$ we have $\phi(x) = w(x) - \psi(x) \ge 0$ and thus *i)* follows from *ii)*.

For the opposite implication we pick some $\phi \in W_0^{1,\infty}(\Omega)$ with $\phi \ge 0$ on Λ. We would like to use *ii)* with $w = \phi + v$. However, this will not work, since in general we do not know if $\phi + v \ge \psi$ in Ω. Let us therefore start with the special case that additionally, for some $\alpha > 0$, we have $\phi \ge 0$ in $\{x \in \Omega \,:\, v(x) \le \psi(x) + \alpha\}$. Then, for $\eta > 0$ small enough, we have $\eta\phi > -\alpha$ in Ω and thus $w := v + \eta\phi \in W^{1,\infty}(\Omega)$ satisfies $w \ge \psi$ in Ω and $w = g$ on $\partial\Omega$. Consequently, by *ii)* we get

$$0 \le \int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla(w-v)\,\mathrm{d}x = \eta \int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla\phi\,\mathrm{d}x.$$

Since $\eta > 0$ we deduce the desired inequality in this special case. In order to treat the general situation we need the following auxiliary claim:

($*$) $\forall \phi$ as in *iii)* $\forall \varepsilon > 0\ \exists \alpha = \alpha(\phi, \varepsilon) > 0:\ \phi \ge -\varepsilon$ on $\{x \in \Omega \,:\, v \le \psi + \alpha\}$.

To prove this claim we just have to note that in $\Lambda^\varepsilon := \{x \in \Omega \,:\, \mathrm{dist}_\Lambda(x) < \frac{\varepsilon}{\mathrm{Lip}(\phi)}\}$ we have $\phi \ge -\varepsilon$. Then, either $\Lambda^\varepsilon = \Omega$, in which case every $\alpha > 0$ does the job, or we can put $\alpha = \frac{1}{2}\min_{\overline{\Omega}\setminus V_\varepsilon}(u - \psi) > 0$. This choice of α will make sure that $\{v \le \psi + \alpha\} \subset \Lambda^\varepsilon$ and thus proves the auxiliary claim.

Let us now fix $\varepsilon_0, \delta > 0$ such that we have $\Lambda^{\varepsilon_0} \subset \Omega_\delta := \{x \in \Omega \,:\, \mathrm{dist}_{\partial\Omega}(x) > \delta\}$. Then for $0 < \varepsilon < \varepsilon_0$ we define

$$\eta_\varepsilon(x) := \begin{cases} \frac{\varepsilon}{\delta}\mathrm{dist}_{\partial\Omega}(x) & x \in \Omega \setminus \Omega_\delta, \\ \varepsilon & x \in \Omega_\delta. \end{cases}$$

As $\eta_\varepsilon \in W_0^{1,\infty}(\Omega)$, also $\phi_\varepsilon := \phi + \eta_\varepsilon \in W_0^{1,\infty}(\Omega)$ and moreover, by the auxiliary claim, ϕ_ε falls into the special case we treated initially and hence we get

$$0 \le \int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla \phi_\varepsilon \, \mathrm{d}x = \int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla \phi \, \mathrm{d}x + \int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla \eta_\varepsilon \, \mathrm{d}x.$$

The desired inequality now follows upon letting $\varepsilon \to 0$ and observing that

$$\left| \int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla \eta_\varepsilon \, \mathrm{d}x \right| \le \frac{\varepsilon}{\delta} \int_{\Omega \setminus \Omega_\delta} 1 \, \mathrm{d}x \to 0 \quad (\varepsilon \to 0).$$

The implication $iii) \implies iv)$ is trivial since $C_c^\infty(\Omega) \subset W_0^{1,\infty}(\Omega)$.

Let us now assume that $iv)$ holds and start by considering in an intermediate step $\phi \in W^{1,\infty}(\Omega) \cap C_c(\Omega)$ with $\phi \ge 0$ on Λ. Extending ϕ outside of Ω by zero, we consider for $\varepsilon > 0$ its mollification $\phi_\varepsilon := (\phi * \rho_\varepsilon)_{|\Omega}$. Note, that since ϕ has compact support in Ω, for ε small enough $\phi_\varepsilon \in C_c^\infty(\Omega)$. Moreover, as $\phi_\varepsilon \to \phi$ in $W^{1,p}(\Omega)$ for all $p < +\infty$ we get that

$$\int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla \phi_\varepsilon \, \mathrm{d}x \to \int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla \phi \, \mathrm{d}x \quad (\varepsilon \to 0). \tag{2.29}$$

However, we cannot yet deduce that the left hand side of the above inequality is non-negative, since we do not know if $\phi_\varepsilon \ge 0$ on Λ. Nevertheless, we can translate ϕ_ε and argue as follows. Let $\delta > 0$ be such that $\Lambda \subset \Omega_\delta$ and choose $\eta \in C_c^\infty(\Omega)$ such that $\eta = 1$ in Ω_δ. Since $\|\phi_\varepsilon - \phi\|_{L^\infty(\Omega)} \le \varepsilon \mathrm{Lip}(\phi)$ we get that $\phi_\varepsilon + \varepsilon \mathrm{Lip}(\phi)\eta \ge 0$ on Λ. Noting also that $\phi_\varepsilon + \varepsilon \mathrm{Lip}(\phi)\eta \in C_c^\infty(\Omega)$, $iii)$ implies now that

$$0 \le \int_\Omega \frac{\nabla v \cdot \nabla(\phi_\varepsilon + \varepsilon \mathrm{Lip}(\phi)\eta)}{\sqrt{1+|\nabla v|^2}} \, \mathrm{d}x = \int_\Omega \frac{\nabla v \cdot \nabla \phi_\varepsilon}{\sqrt{1+|\nabla v|^2}} \, \mathrm{d}x + \varepsilon \mathrm{Lip}(\phi) \int_\Omega \frac{\nabla v \cdot \nabla \eta}{\sqrt{1+|\nabla v|^2}} \, \mathrm{d}x.$$

Since the last term converges to zero as $\varepsilon \to 0$, recalling (2.29) and by the arbitrariness of ϕ we deduce that for every $\phi \in W^{1,\infty}(\Omega) \cap C_c(\Omega)$ with $\phi \ge 0$ on Λ we have

$$0 \le \int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla \phi \, \mathrm{d}x.$$

Ultimately, we consider now the case of a function $\phi \in W_0^{1,\infty}(\Omega)$ with $\phi \ge 0$ on Λ for which we derive the desired inequality by a simple truncation argument from the intermediate case that we just considered. More precisely, we let

$$\eta_\varepsilon(x) := \begin{cases} 0 & x \in \Omega \setminus \Omega_\varepsilon, \\ \frac{1}{\varepsilon} \mathrm{dist}_{\partial \Omega_\varepsilon}(x) & x \in \Omega_\varepsilon \setminus \Omega_{2\varepsilon}, \\ 1 & x \in \Omega_{2\varepsilon}. \end{cases}$$

Consequently, $\phi_\varepsilon := \phi\,\eta_\varepsilon \in W^{1,\infty}(\Omega) \cap C_c(\Omega)$ and for ε small enough we get $\phi_\varepsilon = \phi \geq 0$ on Λ. Therefore we get

$$0 \leq \int_\Omega \frac{\nabla v \cdot \nabla \phi}{\sqrt{1+|\nabla v|^2}} \eta_\varepsilon \,\mathrm{d}x + \int_\Omega \frac{\nabla v \cdot \nabla \eta_\varepsilon}{\sqrt{1+|\nabla v|^2}} \phi \,\mathrm{d}x.$$

As $\varepsilon \to 0$, the first integral converges to $\int_\Omega \frac{\nabla v \cdot \nabla \phi}{\sqrt{1+|\nabla v|^2}}\,\mathrm{d}x$ and the second can be estimate by

$$\left| \int_\Omega \frac{\nabla v \cdot \nabla \eta_\varepsilon}{\sqrt{1+|\nabla v|^2}} \phi \,\mathrm{d}x \right| \leq \int_{\Omega_\varepsilon \setminus \Omega_{2\varepsilon}} |\eta_\varepsilon||\phi| \,\mathrm{d}x. \leq \underbrace{|\Omega_\varepsilon \setminus \Omega_{2\varepsilon}|}_{\leq C\varepsilon} \frac{2}{\varepsilon} \|\phi\|_{L^\infty(\Omega_\varepsilon \setminus \Omega_{2\varepsilon})} \to 0 \quad (\varepsilon \to 0),$$

where we used the fact that $\phi \in W_0^{1,\infty}(\Omega)$ to deduce that $\|\phi\|_{L^\infty(\Omega_\varepsilon \setminus \Omega_{2\varepsilon})} \to 0$. $\square$

3. Distributional Solutions

Let us start this chapter by recalling the scheme which was introduced in section 1.3 and adapted to the setting of Lipschitz functions in section 1.4. We fix $u_0 \in C^2(\overline{\Omega})$ and $\psi \in C^{1,1}(\Omega)$ satisfying $u_0 > \psi$ on $\partial\Omega$, and $u_0 \geq \psi$ in Ω, where Ω is an open, bounded and convex domain with smooth boundary. For a fixed time step size $h > 0$, our aim is to find a sequence $(u_k^h)_{k\in\mathbb{N}}$ in $\mathrm{Lip}(\Omega)$ with the property, that u_k^h minimizes the energy

$$\int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x + \frac{1}{h}\int_\Omega \int_0^{u(x)} \mathrm{sdist}_{u_{k-1}^h}(x,z)\,\mathrm{d}z\,\mathrm{d}x,$$

among all functions u which belong to

$$\mathcal{A} := \{u \in \mathrm{Lip}(\Omega) : u_{|\partial\Omega} = u_{0|\partial\Omega},\, u \geq \psi \ \text{ in } \ \Omega\},$$

and where, as in Definition 1.2.2, sdist_v denotes the signed distance to the subgraph of a function v. Such a sequence $(u_k^h)_{k\in\mathbb{N}}$ will be called an *approximate flow* (for the time step size h). In order to pass into the limit $h \to 0$, we would need uniform (in h and k) estimates on the Lipschitz constants of the u_k^h's. In this chapter we will now in a first step show how to apply our abstract results from Chapter 2 to derive such a bound. Subsequently, using the control on the Lipschitz constant, we can also derive a couple of important properties of the time discrete evolution, which eventually allow us to derive the existence of so called *distributional solutions* to the parabolic obstacle problem. Finally we will discuss the asymptotic limit of these solutions and show that they converge to constrained minimal surfaces.

3.1. The Uniform Lipschitz Estimate

The strategy will be to use Theorem 2.2.8 iteratively for the construction of the approximate flow. We recall that in Chapter 2, H was used to denote the forcing term which will henceforth be of the form $H(x,z) := \frac{1}{h}\mathrm{sdist}_u(x,z)$, for varying $u : \Omega \to \mathbb{R}$, Lipschitz. Let us also note, that c_0 denotes the quantity $\inf_{x\in\Omega} \operatorname{ess\,inf}_{z'\in\mathbb{R}} \partial_z H(x,z')$ which was assumed to be positive (and will be deduced in Corollary 3.1.2 below for the case under consideration). As we have seen, one part of the estimate on the Lipschitz constant provided by Theorem 2.2.8 is given by $c_0^{-1}\|\nabla_x H\|_{L^\infty(\Omega\times\mathbb{R})}$. We consider for the moment just points $(x,z) \in \Omega \times \mathbb{R}$ sufficiently close to the graph of u_0, where sdist_{u_0} is smooth (cf. Proposition 1.2.6). Then for some $y \in \Omega$ we have

$$\nabla_{\mathbb{R}^{n+1}}\mathrm{sdist}_{u_0}(x,z) = \frac{1}{h}\frac{(-\nabla u_0(y),1)}{\sqrt{1+|\nabla u_0(y)|^2}},$$

and consequently setting $L := \mathrm{Lip}(u)$, for such points we can estimate

$$\partial_z \mathrm{sdist}_{u_0}(x,z) \geq \frac{1}{h}\frac{1}{\sqrt{1+L^2}} > 0 \quad \text{and} \quad |\nabla_x \mathrm{sdist}_{u_0}(x,z)| \leq \frac{1}{h}\frac{L}{\sqrt{1+L^2}}.$$

As we will show in the next proposition, these estimates can be extended to all of $\Omega \times \mathbb{R}$ and thus we will be able to deduce $c_0^{-1}\|\nabla_x H\|_{L^\infty(\Omega\times\mathbb{R})} \leq \mathrm{Lip}(u)$.

Proposition 3.1.1. *Let $\Omega \subset \mathbb{R}^n$ be open and convex. Suppose $u : \Omega \to \mathbb{R}$ is L-Lipschitz and denote by E its (closed) subgraph i.e.*

$$E := \{(x,z) \in \overline{\Omega} \times \mathbb{R} : z \leq u(x)\}.$$

Then, for any $x_0 \in \overline{\Omega}$ and any $z_0 \in \mathbb{R}$ we have:

i) $\mathrm{sdist}_u(x_0,z_2) - \mathrm{sdist}_u(x_0,z_1) \geq \frac{1}{\sqrt{1+L^2}}|z_1 - z_2|$ *for all $z_1 \leq z_2 \in \mathbb{R}$.*

ii) $|\mathrm{sdist}_u(x_1,z_0) - \mathrm{sdist}_u(x_2,z_0)| \leq \frac{L}{\sqrt{1+L^2}}|x_1 - x_2|$ *for all $x_1, x_2 \in \overline{\Omega}$.*

Proof. *i)* Fix any $x_0 \in \overline{\Omega}$ and let us first discuss the case $z_1 = u(x_0)$. Since u is Lipschitz, its subgraph E is contained in the set $C := \{(x,z) \in \overline{\Omega} \times \mathbb{R} : z \leq z_1 + L|x - x_0|\}$ and hence by Lemma 1.2.1 *i)* it suffices to estimate the signed distance to C. In fact, for any $z_2 \geq z_1$ one can easily compute that $\mathrm{sdist}_C(x_0,z_2) \geq \frac{1}{\sqrt{1+L^2}}|z_2 - z_1|$ and the desired estimate follows. Next, we argue, why all the other cases can be reduced to the following inequality:

$$\mathrm{sdist}_u(x_0,z_2) - \mathrm{sdist}_u(x_0,z_1) \geq \frac{1}{\sqrt{1+L^2}}|z_1 - z_2| \quad \forall z_2 \geq z_1 > u(x_0). \tag{3.1}$$

Indeed, by definition of the signed distance, $\mathrm{sdist}_u(x,z) = -\mathrm{sdist}_{(\overline{\Omega}\times\mathbb{R})\setminus E}(x,z)$ so that the case $z_1 \leq z_2 \leq u(x)$ reduces to (3.1) by simply reflecting along a horizontal hyperplane. In the remaining case that $z_1 < u(x_0) < z_2$ we introduce $\bar{z} := u(x_0)$ and apply twice the already discussed cases to obtain:

$$\begin{aligned}\mathrm{sdist}_u(x_0,z_2) - \underbrace{\mathrm{sdist}_u(x_0,\bar{z}) + \mathrm{sdist}_u(x_0,\bar{z})}_{=0} &- \mathrm{sdist}_u(x_0,z_1) \\ &\geq \frac{1}{\sqrt{1+L^2}}|z_2 - \bar{z}| + \frac{1}{\sqrt{1+L^2}}|\bar{z} - z_1| = \frac{1}{\sqrt{1+L^2}}|z_2 - z_1|.\end{aligned}$$

Now we will establish equation (3.1). For fixed $z_2 \geq z_1 > u(x_0)$ we choose $\bar{x} \in \overline{\Omega}$ such that

$$0 < d := \mathrm{sdist}_u(x_0,z_1) = |(x_0,z_1) - (\bar{x},u(\bar{x}))|.$$

The possibility that $u(\bar{x}) > z_1$ can be ruled out by only using the continuity of u, cf. also Figure 3.1. Indeed, recalling that $u(x_0) < z_1$, the intermediate value theorem and the convexity of Ω would imply the existence of a point $y \in \overline{\Omega}$ in the interior of the segment $[x_0;\bar{x}]$ with $u(y) = z_1$. This would clearly contradict the fact that the distance is realized at $(\bar{x}, u(\bar{x}))$. Using additionally the fact that u is Lipschitz we can even estimate the distance from $\bar{x}$ to x_0. This will be the key observation. In order not to contradict our assumption, the graph of u above the set $B_r(x_0; \mathbb{R}^n) \cap \overline{\Omega}$, where $r := |x_0 - \bar{x}|$, must stay below the graph of the function $f(x) = z_1 - \sqrt{d^2 - (x_0 - x)^2}$. However, since $u(\bar{x}) = f(\bar{x})$ this can only be true if $L \geq |\nabla f(\bar{x})| = \frac{r}{\sqrt{d^2 - r^2}}$, from which we derive the necessary condition

$$\frac{Ld}{\sqrt{1+L^2}} \geq r. \tag{3.2}$$

Using once more the fact that u is Lipschitz we can therefore conclude that E has to be contained in G, by which we denote the (closed) subgraph of the function g where for $x \in \mathbb{R}^n$

$$g(x) := \begin{cases} f(x) & \text{if } |x - x_0| \leq \frac{Ld}{\sqrt{1+L^2}}, \\ L|x - x_0| + z_1 - d\sqrt{1+L^2} & \text{else.} \end{cases}$$

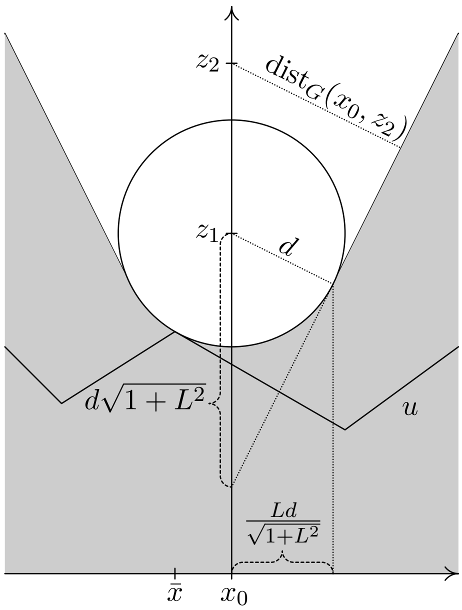

Figure 3.1.: The auxiliary set G (shaded).

The distance of (x_0, z_2) to G can now simply be computed. As $z_2 > z_1$ we can easily rule out the possibility that this distance is obtained at a point $(x, g(x))$ with $0 < |x - x_0| \leq Ld(1 + L^2)^{-\frac{1}{2}}$. Indeed, to see this, it suffices to observe that for the function $\phi(x) := (x_0 - x)^2 + (z_2 - f(x))^2$ we have $\nabla\phi(x) = 0$ if and only if $x = x_0$. Assuming for the moment that the distance $\mathrm{sdist}_G(x_0, z_2)$ is also not obtained at $(x_0, z_1 - d)$, we can use the relation

$$\frac{\mathrm{sdist}_G(x_0, z_2)}{d} = \frac{z_2 - z_1 + d\sqrt{1+L^2}}{d\sqrt{1+L^2}},$$

which is just an instance of the intercept theorem, see again Figure 3.1. Subtracting 1 on both sides and then multiplying by d we get

$$\operatorname{sdist}_G(x_0, z_2) - d = \frac{z_2 - z_1}{\sqrt{1+L^2}}.$$

Firstly, we can now a posteriori rule out that the distance of (x_0, z_2) is obtained at $(x_0, z_1 - d)$ (as the distance to this point is bigger, namely $z_2 - z_1 + d$). Secondly, we can now deduce (3.1) by the preceding computation and Lemma 1.2.1 *i)* since $E \subset G$.

ii) One can argue very similar to the previous claim. First of all, the case where either (x_1, z_0) or (x_2, z_0) lie exactly on the graph follows easily from the fact that if for instance $u(x_1) = z_0$ then the graph of u is contained in $\{(x,z) \in \overline{\Omega} \times \mathbb{R} : |z - z_0| \leq L|x - x_1|\}$ which allows us to estimate easily the signed distance of (x_2, z_0) to the subgraph of u. Furthermore, as before, all the remaining cases can be reduced to the one where both (x_1, z_0) and (x_2, z_0) lie above the graph of u. Without loss of generality we assume that $0 < d := \operatorname{sdist}_u(x_1, z_0) \leq \operatorname{sdist}_u(x_2, z_0)$ and we let $\bar{x} \in \Omega$ again be such that $\operatorname{sdist}_u(x_1, z_0) = |(x_1, z_0) - (\bar{x}, u(\bar{x}))|$. As before, we know that the subgraph of u has to contain the set $F := \{(x,z) \in \overline{\Omega} \times \mathbb{R} : z \leq u(\bar{x}) - L|x - \bar{x}|\}$ which allows us to estimate by Lemma 1.2.1

$$\operatorname{sdist}_u(x_2, z_0) \leq \operatorname{dist}_F(x_2, z_0).$$

Among all the possible locations of $\bar{x}$, the distance of (x_2, z_0) to F can be estimated

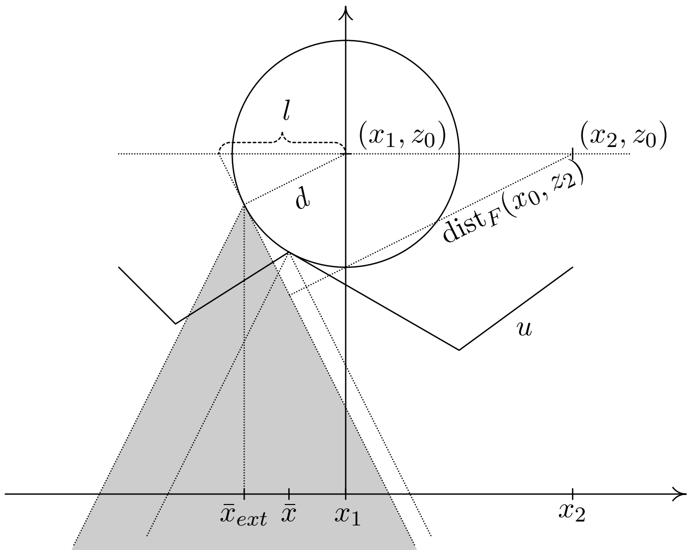

Figure 3.2.: The auxiliary set F for generic $\bar{x}$ and for $\bar{x}_{ext}$ (shaded).

from above by henceforth considering the case

$$\bar{x} = \bar{x}_{ext} := x_1 + \frac{Ld}{\sqrt{1+L^2}} \frac{x_1 - x_2}{|x_1 - x_2|},$$

(here we used the same argument about the distance $|\bar{x} - x_1|$ as in *i)*) cf. Figure 3.1. All that remains now is a computation analogous to the one in the previous claim. Indeed, another instance of the intercept theorem (see again Figure 3.1) yields:

$$\frac{\mathrm{sdist}_F(x_2, z_0)}{d} = \frac{|x_2 - x_1| + ld}{ld},$$

where $l := \frac{\sqrt{1+L^2}}{L}$. From this relation we easily deduce

$$\mathrm{sdist}_F(x_2, z_0) - d \leq \frac{L}{\sqrt{1+L^2}}|x_2 - x_1|,$$

and the claim follows again by Lemma 1.2.1 as $F \subset E$. □

Corollary 3.1.2. *Assume $u : \Omega \to \mathbb{R}$ is L-Lipschitz and for $(x, z) \in \Omega \times \mathbb{R}$ we set $H(x, z) := \frac{1}{h}\mathrm{sdist}_u(x, z)$. Then H is Lipschitz,*

$$c_0 := \inf_{x \in \Omega} \operatorname*{ess\,inf}_{z' \in \mathbb{R}} \partial_z H(x, z') > 0, \quad \textit{and} \quad \frac{\|\nabla_x H\|_{L^\infty(\Omega\times\mathbb{R})}}{c_0} \leq L.$$

Proof. To see that H is Lipschitz, we just recall Definition 1.2.2 and Proposition 1.2.3. The two inequalities are direct consequences of the previous proposition. □

As already mentioned in the beginning of this section, we would like to use Theorem 2.2.8 iteratively. However, the estimate given by the theorem depends also on the barriers v^+, v^-, which in turn are constructed from the initial (and boundary) data u_0. Note, that for this purpose, u_0 was assumed to be C^2-regular, while we do not expect our iterative solutions u_k to have such regularity. We need then to show that this part of the estimate remains unchanged in the iteration of the estimate.

Theorem 3.1.3. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex with smooth boundary and let $u_0 \in C^2(\overline{\Omega})$, $\psi \in C^{1,1}(\Omega)$ such that $u_0 > \psi$ on $\partial\Omega$ and $u_0 \geq \psi$ in Ω. Furthermore, we fix $h > 0$. Then there exists a unique sequence of Lipschitz functions $(u_k^h)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that for $k \geq 1$, u_k^h minimizes the energy*

$$E_{k-1}(u) := \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\,\mathrm{d}x + \frac{1}{h}\int_\Omega\int_0^{u(x)} \mathrm{sdist}_{u_{k-1}^h}(x, z)\,\mathrm{d}z\mathrm{d}x,$$

among all $u \in \mathcal{A}$. Moreover, we have

$$\mathrm{Lip}(u_k^h) \leq L,$$

where the constant L depends only on Ω, ψ and u_0 but not on h or k.

Proof. To obtain u_1^h, we apply Theorem 2.2.8 directly to the problem of minimizing E_0 on $\mathcal{A}$. Setting $H = \frac{1}{h}\mathrm{sdist}_{u_0}$, by the previous corollary we know that H is Lipschitz and that $c_0 > 0$ and thus (2.11) holds. Hence we just have to verify (2.13), i.e. $H(x, u_0(x)) = 0$ for all $x \in \Omega$. However, this holds trivially by definition of H. Thus, by Theorem 2.2.8 and again by the previous corollary, we obtain the estimate

$$\mathrm{Lip}(u_1^h) \leq \max\{C(\Omega, u_0, \psi), \mathrm{Lip}(\psi), \mathrm{Lip}(u_0)\} =: L.$$

Let us now denote by v^+ and v^- the barriers constructed as in the proof of Theorem 2.2.8 to obtain the solution u_1^h. We claim that we can recycle them as barriers for u_2^h. To see this, we first of all recall that to check that v^+ is an upper barrier, the main point was to establish inequality (2.15). In the case $H = \frac{1}{h}\mathrm{sdist}_{u_0}$ this was trivially implied by ensuring that $v_0^+ \geq u_0$ in $\Omega \setminus \Omega_t$ for some $t > 0$. Consequently, to show that v^+ is also an upper barrier for u_2^h, i.e. for the problem of minimizing E_1, it suffices to establish that

$$v^+ \geq u_1^h \quad \text{in } \Omega \setminus \Omega_t, \tag{3.3}$$

$$v^+ \geq \sup_{\Omega} u_1^h \quad \text{on } \partial\Omega_t. \tag{3.4}$$

However, since v^+ was an upper barrier for minimizing E_0, (3.3) holds trivially (by the comparison principle, Proposition 2.2.4). One can argue similarly for the lower barrier v^-. Moreover, by construction $v^+ \geq M = \sup_\Omega u_1^h$ *on* $\partial\Omega_t$ and by Lemma 2.2.6 we know that $u_1^h \leq M$ in Ω so that also (3.4) holds and v^+ is indeed an upper barrier for u_2^h. Having therefore established barriers for the problem of minimizing E_1, we can again apply Proposition 2.2.7 to deduce a boundary gradient estimate and then continue to argue as in the proof of Theorem 2.2.8 to get the existence of u_2^h, which minimize E_1 on $\mathcal{A}$. Moreover, again due to Corollary 3.1.2 we get

$$\frac{||\nabla_x \mathrm{sdist}_{u_1^h}(x,z)||_{L^\infty(\Omega\times\mathbb{R})}}{\inf_{x\in\Omega} \operatorname{ess\,inf}_{z'\in\mathbb{R}} \partial_z \mathrm{sdist}_{u_1^h}(x,z')} \leq \mathrm{Lip}(u_1^h) = L \quad \forall x \in \Omega,$$

so that due to Proposition 2.2.5 we get

$$\mathrm{Lip}(u_2^h) \leq \max\{L, \mathrm{Lip}(\psi)\} = L.$$

Since this way of arguing can now be iterated, we conclude the proof. □

3.2. Properties of Approximate Flows

In this section we collect some important results on this time discrete evolution. Let us start by fixing the notation.

Definition 3.2.1. From now on, for $h > 0$, we denote by $u^h : \overline{\Omega} \times [0, +\infty[\to \mathbb{R}$ the time discrete approximate flow obtained from Theorem 3.1.3, this means that we set

$$u^h(x,t) := u_k^h(x) \quad \textit{for } k \in \mathbb{N}_0 \textit{ such that } t \in [kh, (k+1)h[,$$

where u_k^h is the sequence obtained in Theorem 3.1.3.

Remark 3.2.1. Since all the u_k^h's are continuous, it is easy to see that u^h is *continuous from the right in time*, in the sense that whenever $x_k \to x$ in Ω and $t_k \to t$ in $\mathbb{R}$ with $t_k \geq t$ then it holds

$$\lim_{k\to+\infty} u^h(x_k, t_k) \to u^h(x,t).$$

Let us now start with the *(discrete) dissipation inequality*. To simplify the notation, at many places, we will henceforth suppress the superscript in the expression u_k^h as $h > 0$ is often considered to be an arbitrary but fixed parameter.

Lemma 3.2.1. *For every $k \geq 1$ we have the so called (discrete) dissipation inequality*

$$\int_\Omega \sqrt{1+|\nabla u_k|^2}\,\mathrm{d}x + \frac{1}{h}\int_\Omega\int_{u_{k-1}(x)}^{u_k(x)} \mathrm{sdist}_{u_{k-1}}(x,z)\,\mathrm{d}z\,\mathrm{d}x \leq \int_\Omega \sqrt{1+|\nabla u_{k-1}|^2}\,\mathrm{d}x.$$

Furthermore, for every $k \geq 1$:

$$\int_\Omega \sqrt{1+|\nabla u_k|^2}\,\mathrm{d}x \leq \int_\Omega \sqrt{1+|\nabla u_0|^2}\,\mathrm{d}x, \tag{3.5}$$

$$\frac{1}{h}\int_\Omega\int_{u_{k-1}(x)}^{u_k(x)} \mathrm{sdist}_{u_{k-1}}(x,z)\,\mathrm{d}z\,\mathrm{d}x \leq \int_\Omega \sqrt{1+|\nabla u_0|^2}\,\mathrm{d}x. \tag{3.6}$$

Proof. We start by splitting the distance term as follows:

$$\begin{aligned}\int_\Omega\int_0^{u_k(x)} &\mathrm{sdist}_{u_{k-1}}(x,z)\,\mathrm{d}z\,\mathrm{d}x \\ &= \int_\Omega\int_0^{u_{k-1}(x)} \mathrm{sdist}_{u_{k-1}}(x,z)\,\mathrm{d}z\,\mathrm{d}x + \int_\Omega\int_{u_{k-1}(x)}^{u_k(x)} \mathrm{sdist}_{u_{k-1}}(x,z)\,\mathrm{d}z\,\mathrm{d}x. \end{aligned}\tag{3.7}$$

Observe, that the first term is independent of u_k and the second one is nonnegative. Indeed,

$$\begin{aligned}\int_\Omega\int_{u_{k-1}(x)}^{u_k(x)} &\mathrm{sdist}_{u_{k-1}}(x,z)\,\mathrm{d}z\,\mathrm{d}x \\ &= \int_{\{u_k\geq u_{k-1}\}}\int_{u_{k-1}(x)}^{u_k(x)} \underbrace{\mathrm{sdist}_{u_{k-1}}}_{\geq 0}\,\mathrm{d}z\,\mathrm{d}x - \int_{\{u_k< u_{k-1}\}}\int_{u_k(x)}^{u_{k-1}(x)} \underbrace{\mathrm{sdist}_{u_{k-1}}}_{\leq 0}\,\mathrm{d}z\,\mathrm{d}x \geq 0.\end{aligned}$$

Hence, u_k is also a minimizer of the nonnegative energy

$$\widehat{E}_k(u) := \int_\Omega \left(\sqrt{1+|\nabla u(x)|^2} + \frac{1}{h}\int_{u_{k-1}(x)}^{u(x)} \mathrm{sdist}_{u_{k-1}}(x,z)\mathrm{d}z\right)\mathrm{d}x.$$

The dissipation inequality now follows from $\widehat{E}_k(u_k) \leq \widehat{E}_k(u_{k-1})$. (3.5) and (3.6) are obtained by iterating the dissipation inequality. $\square$

We collect two easy lemmas on Lipschitz functions which turn out to be very useful.

Lemma 3.2.2. *Let $u \in \mathrm{Lip}(\Omega)$ with Lipschitz constant $L \geq 0$, $\Omega \subset \mathbb{R}^n$. Fix a ball $B_r(x_0) \subset \Omega$. Then for $\delta := (\sqrt{1+L^2})^{-1}$ we have:*

$$\mathrm{graph}(u_{|B_{\delta r}(x_0)}) \subset B_r((x_0, u(x_0)); \mathbb{R}^{n+1}).$$

Proof. Let $y \in B_{\delta r}(x_0)$. Then we get

$$\begin{aligned}|(x_0, u(x_0)) - (y, u(y))| &= \sqrt{|x_0 - y|^2 + |u(x_0) - u(y)|^2} \\ &\le \sqrt{1 + L^2}|x_0 - y| < r.\end{aligned}$$

□

Lemma 3.2.3. *For $L \ge 0$ let u and w be two L-Lipschitz functions defined on $\Omega \subset \mathbb{R}^n$. Then we have*

$$|u(x) - w(x)| \le \sqrt{1 + L^2}\,|\mathrm{sdist}_w(x, u(x))| \quad \forall x \in \Omega.$$

Proof. Fix $x_0 \in \Omega$ and let $y_0 \in \overline{\Omega}$ such that

$$d := |\mathrm{sdist}_w(x_0, u(x_0))| = |(x_0, u(x_0)) - (y_0, w(y_0))|.$$

Without loss of generality we assume that $w(x_0) \ge u(x_0)$, the other case being treated analogous. Then we have $u(x_0) \le w(y_0) \le w(x_0)$ and hence

$$\begin{aligned}|u(x_0) - w(x_0)| &= |w(x_0) - w(y_0)| + |w(y_0) - u(x_0)| \\ &\le L|x_0 - y_0| + |w(y_0) - u(x_0)|.\end{aligned}$$

Now recall from the proof of Proposition 3.1.1 (see (3.2)) that

$$|x_0 - y_0| \le \frac{L}{\sqrt{1 + L^2}} d.$$

Let now $0 \le \alpha \le L$ be such that $|x_0 - y_0| = \frac{\alpha}{\sqrt{1+\alpha^2}} d$ and note that this implies

$$|w(y_0) - u(x_0)| = \frac{d}{\sqrt{1 + \alpha^2}},$$

where we used the relation $d^2 = |x_0 - y_0|^2 + |w(y_0) - u(x_0)|^2$. Since $0 \le \alpha \le L$ we conclude

$$|u(x_0) - w(x_0)| \le \frac{(L\alpha + 1)d}{\sqrt{1 + \alpha^2}} \le \sqrt{1 + L^2} d.$$

□

Proposition 3.2.4. *(L^∞-bound) Let $(u_k^h)_{k \in \mathbb{N}}$, $h > 0$ and L be as in Theorem 3.1.3. Then there exists a constant $\gamma = \gamma(n, L) > 0$, such that*

$$\sup_{x \in \Omega} \left|\mathrm{sdist}_{u_{k-1}^h}(x, u_k^h(x))\right| \le \gamma\sqrt{h} \quad \forall k \ge 1.$$

Proof. Of course, it suffices to consider the case $k = 1$ and to simplify the notation we drop again the superscript h. Let $x_0 \in \Omega$ be a point of maximal distance i.e. such that

$$|\mathrm{sdist}_{u_0}(x_0, u_1(x_0))| = \sup_{x \in \Omega} |\mathrm{sdist}_{u_0}(x, u_1(x))|.$$

Let $\kappa > 0$ be such that $|\mathrm{sdist}_{E_0}(x_0, u_1(x_0))| = \kappa\sqrt{h}$. The idea is to construct a competitor which (in a small neighborhood of x_0) is a bit closer to u_0. Using then the minimality of u_1 we can bound κ from above by a constant depending only on n and L. Without loss of generality $u_1(x_0) < u_0(x_0)$, the other case being treated similarly. We then define $\widetilde{u} := \max\{u_1, g\}$ where

$$g(x) := u_1(x_0) + \frac{L}{\sqrt{1+L^2}} \frac{\kappa\sqrt{h}}{2} - 2L|x - x_0|.$$

Let us note that since u_1 has to stay between

$$f_1(x) = u_1(x_0) - L|x - x_0|,$$

and

$$f_2(x) = u_1(x_0) + L|x - x_0|,$$

we know that on $\partial B_\rho(x_0)$, for $\rho = \delta\frac{\kappa\sqrt{h}}{2}$, we have $g \leq u_1$, while in some neighborhood of x_0, (at least in $B_{\frac{1}{3}\rho}(x_0)$) g is strictly bigger than u_1. Indeed, elementary calculations show that

$$\int_{B_{\frac{1}{3}\rho}(x_0)} \int_{u_1(x)}^{g(x)} \mathrm{d}z\mathrm{d}x \geq C(n, L)(\rho)^{n+1}. \tag{3.8}$$

By minimality of u_1 we deduce

$$\int_{B_{\frac{1}{3}\rho}(x_0)} \sqrt{1 + |\nabla u_1|^2}\,\mathrm{d}x + \frac{1}{h}\int_{B_{\frac{1}{3}\rho}(x_0)} \int_0^{u_1} \mathrm{sdist}_{u_0}(x, z)\,\mathrm{d}z\,\mathrm{d}x$$
$$\leq \int_{B_{\frac{1}{3}\rho}(x_0)} \sqrt{1 + |\nabla \widetilde{u}|^2}\,\mathrm{d}x + \frac{1}{h}\int_{B_{\frac{1}{3}\rho}(x_0)} \int_0^{\widetilde{u}} \mathrm{sdist}_{u_0}(x, z)\,\mathrm{d}z\,\mathrm{d}x,$$

rearranging terms

$$\frac{1}{h}\int_{B_{\frac{1}{3}\rho}(x_0)} \int_{\widetilde{u}}^{u_1} \mathrm{sdist}_{u_0}(x, z)\,\mathrm{d}z\,\mathrm{d}x \leq \int_{B_{\frac{1}{3}\rho}(x_0)} \sqrt{1 + |\nabla \widetilde{u}|^2} - \sqrt{1 + |\nabla u_1|^2}\,\mathrm{d}x.$$

Using (3.8), $\mathrm{Lip}(\widetilde{u}) \leq 2L$ and

$$\mathrm{sdist}_{E_0}(x, z) < -\frac{\kappa\sqrt{h}}{2} \quad \forall (x, z) \in B_{\frac{\kappa\sqrt{h}}{2}}((x_0, u_1(x_0)); \mathbb{R}^{n+1}),$$

we get

$$\frac{1}{h}\frac{\kappa\sqrt{h}}{2} C(n, L)\rho^{n+1} \leq (\sqrt{1 + 4L^2} - 1)C(n)\rho^n.$$

Recalling $\rho = \delta \frac{\kappa\sqrt{h}}{2}$, we finally get

$$\kappa^2 \leq C(n, L),$$

and the proposition follows upon setting $\gamma = \sqrt{C(n, L)}$. ☐

Lemma 3.2.5. *(L^1-estimate) Let $l \geq 0$. Then for every $k \geq 1$ we have*

$$\int_\Omega |u_k - u_{k-1}| \mathrm{d}x \leq l|\Omega| + \frac{C(L)}{l} \int_\Omega \int_{u_{k-1}(x)}^{u_k(x)} \mathrm{sdist}_{u_{k-1}}(x, z)\, \mathrm{d}z\, \mathrm{d}x.$$

Proof. Again, it suffices to prove the case $k = 1$. We have

$$\begin{aligned} \int_\Omega |u_1 - u_0| \mathrm{d}x &\leq l|\Omega| + \int_\Omega \max\{0, |u_1 - u_0| - l\} \mathrm{d}x \\ &= l|\Omega| + \int_\Omega \left| \int_{u_0(x)}^{u_1(x)} \chi_{\mathbb{R}\setminus[u_0(x)-l, u_0(x)+l]}(z) \mathrm{d}z \right| \mathrm{d}x. \end{aligned} \tag{3.9}$$

Here it is necessary to take the absolute value since if $u_1(x) < u_0(x)$ we have

$$\int_{u_0(x)}^{u_1(x)} \chi_{\mathbb{R}\setminus[u_0(x)-l, u_0(x)+l]}(z)\, \mathrm{d}z = -\max\{0, |u_1 - u_0| - l\}.$$

However, we can use the sign convention for one-dimensional integrals together with the relation

$$\chi_{\mathbb{R}\setminus[u_0(x)-l, u_0(x)+l]}(z) \leq \left| \frac{z - u_0(x)}{l} \right| \quad \forall z \in \mathbb{R},$$

to deduce

$$\left| \int_{u_0(x)}^{u_1(x)} \chi_{\mathbb{R}\setminus[u_0(x)-l, u_0(x)+l]}(z)\, \mathrm{d}z \right| \leq \int_{u_0(x)}^{u_1(x)} \frac{z - u_0(x)}{l}\, \mathrm{d}z. \tag{3.10}$$

By part i) of Proposition 3.1.1 we get for every $x \in \Omega$ and every $z \in \mathbb{R}$

$$\begin{aligned} z - u_0(x) &\leq C(L) \mathrm{sdist}_{u_0}(x, z) \quad \text{if } u_0(x) \leq u_1(x), \\ -(z - u_0(x)) &\leq -C(L) \mathrm{sdist}_{u_0}(x, z) \quad \text{if } u_0(x) \geq u_1(x). \end{aligned}$$

Hence we get for every $x \in \Omega$

$$\int_{u_0(x)}^{u_1(x)} \frac{z - u_0(x)}{l}\, \mathrm{d}z \leq C(L) \int_{u_0(x)}^{u_1(x)} \frac{\mathrm{sdist}_{u_0}(x, z)}{l}\, \mathrm{d}z. \tag{3.11}$$

Combining (3.9), (3.10) and (3.11) gives the desired estimate. ☐

Proposition 3.2.6. *($C^{1/2}$ in time) There exists a constant $C = C(\Omega, u_0, L)$ such that for every $h > 0$*

$$\int_\Omega \left| u^h(\cdot, t) - u^h(\cdot, s) \right| \mathrm{d}x \leq C|s - t|^{\frac{1}{2}} \quad \forall s, t \geq 0.$$

Proof. Of course, it is enough to consider only the case $s - t \geq h$. Let $j, k \in \mathbb{N}$ hence be such that $t \in [jh, (j+1)h[$ and $s \in [(j+k)h, (j+k+1)h[$. Using the triangle inequality we get

$$\int_\Omega \Big| u^h(\cdot, t) - u^h(\cdot, s) \Big| \,\mathrm{d}x \leq \sum_{m=1}^{k} \int_\Omega \Big| u^h(\cdot, (j+m)h) - u^h(\cdot, (j+m-1)h) \Big| \,\mathrm{d}x.$$

We apply Lemma 3.2.5 with $l = \frac{h}{\sqrt{s-t}}$ to each term on the right hand side to get

$$\int_\Omega \Big| u^h(\cdot, t) - u^h(\cdot, s) \Big| \,\mathrm{d}x \leq \sum_{m=1}^{k} \Bigg[\frac{h}{\sqrt{s-t}} |\Omega| \\ + C(L) \frac{\sqrt{s-t}}{h} \int_\Omega \int_{u^h(x,(j+m-1)h)}^{u^h(x,(j+m)h)} \mathrm{sdist}_{u^h(\cdot,(j+m-1)h)}(x, z) \,\mathrm{d}z \,\mathrm{d}x \Bigg].$$

Next, we use the Lemma 3.2.1 to estimate the second term further and obtain

$$\int_\Omega \Big| u^h(\cdot, t) - u^h(\cdot, s) \Big| \leq \Bigg[\frac{kh}{\sqrt{s-t}} |\Omega| \\ + \sum_{m=j+1}^{j+k} C(L) \sqrt{s-t} \int_\Omega \sqrt{1 + |\nabla u^h(x, (m-1)h)|^2} - \sqrt{1 + |\nabla u^h(x, mh)|^2} \,\mathrm{d}x \Bigg].$$

Since $kh \leq |s - t|$ and using the cancellations in the telescopic sum we get

$$\int_\Omega \Big| u^h(\cdot, t) - u^h(\cdot, s) \Big| \,\mathrm{d}x \\ \leq \sqrt{s-t} |\Omega| + C(L) \sqrt{s-t} \int_\Omega \sqrt{1 + |\nabla u^h(\cdot, t)|^2} - \sqrt{1 + |\nabla u^h(\cdot, s)|^2} \,\mathrm{d}x.$$

The proposition follows now from

$$\int_\Omega \sqrt{1 + |\nabla u^h(\cdot, t)|^2} - \sqrt{1 + |\nabla u^h(\cdot, s)|^2} \,\mathrm{d}x \leq \int_\Omega \sqrt{1 + |\nabla u_0|^2} \,\mathrm{d}x,$$

which is a consequence of (3.5). □

Similarly to Lemma 2.1 in [96] we can prove an L^2-bound in spacetime for the *discrete velocity*. Let us make this notion more precise.

Definition 3.2.2. For $h > 0$ we define the discrete velocity at $(x, t) \in \Omega \times]h, \infty[$ as

$$v^h(x, t) := \frac{1}{h} \mathrm{sdist}_{u^h(\cdot, t-h)}(x, u^h(x, t)).$$

Proposition 3.2.7. *There exists a constant $C > 0$, not depending on h such that*

$$\int_h^{+\infty} \int_\Omega \Big(v^h(x, t) \Big)^2 \,\mathrm{d}x \,\mathrm{d}t \leq C.$$

Proof. Let $t \in [h, +\infty[$. For $l \in \mathbb{Z}$ we set

$$K_t(l) := \{x \in \Omega : 2^l < \left|v^h(x,t)\right| \leq 2^{l+1}\}.$$

Let l_0 be the smallest integer such that $2^{l_0} > \frac{\gamma}{\sqrt{h}}$, where γ is the constant from Proposition 3.2.4, so that we have

$$\Omega = \bigcup_{-\infty < l < l_0} K_t(l).$$

Let us introduce the following notation. For $x \in \Omega$, $z \in \mathbb{R}$ and $r > 0$ with $B_r(x) \subset \Omega$ we consider the adapted cylinder

$$C_r(x,z) = B_{\delta r}(x) \times [z - L\delta r, z + L\delta r] \subset \mathbb{R}^{n+1},$$

where δ is the geometric constant from Lemma 3.2.2. Note, that δ was defined such that the graph of $v_{|B_{\delta r}(x)}$ will be contained in $C_r(x, v(x))$, where v is a generic function. Let $x \in K_t(l)$. First of all, let us observe that while the graph of $u^h(\cdot, t)$ is contained in $C_{2^l h}(x, u^h(x,t))$, by definition of $K_t(l)$, the graph of $u^h(\cdot, t-h)$ cannot enter this cylinder. Hence, in particular $B_{\delta 2^l h}(x) \subset \Omega$. Consequently, there exists a constant $C(n, L)$ such that with $E_t := E_t^h := \{(x,z) \in \Omega \times \mathbb{R} :\ z < u^h(x,t)\}$ we get

$$\left|E_t \cap C_{2^{l-1}h}(x, u^h(x,t))\right| \geq C(n,L)(2^{l-1}h)^{n+1}.$$

Note, that since we have $x \in K_t(l)$,

$$2^{l-1} \leq \frac{\left|\mathrm{sdist}_{u^h(\cdot,t-h)}(y,z)\right|}{h} \leq 4 \cdot 2^{l-1} \quad \forall (y,z) \in C_{2^{l-1}h}(x, u^h(x,t)).$$

Consequently, on the one hand, we have

$$\int_{C_{2^{l-1}h}(x,u^h(x,t)) \cap (E_t \Delta E_{t-h})} \frac{\left|\mathrm{sdist}_{u^h(\cdot,t-h)}\right|}{h} \geq C(n,L)(2^{l-1}h)^{n+1} 2^{l-1},$$

while on the other hand

$$\int_{B_{\delta 2^{l-1}h}(x)} (v^h(x,t))^2 \mathrm{d}x \leq C(n,L)(2^{l-1})^2 (2^{l-1}h)^n.$$

Hence, combining the last two estimates, for every $x \in K_t(l)$ we have

$$\int_{B_{\delta 2^{l-1}h}(x)} (v^h(x,t))^2 \,\mathrm{d}x \leq \frac{C(n,L)}{h} \int_{C_{2^{l-1}h}(x,u^h(x,t)) \cap (E_t \Delta E_{t-h})} \frac{\left|\mathrm{sdist}_{u^h(\cdot,t-h)}\right|}{h}.$$

Since $B_{\delta 2^{l-1}}(x))_{x \in K_t(l)}$ covers $K_t(l)$, we can apply Besicovitch's covering theorem to get $N = N(n)$ (finite) subsets of points in $K_t(l)$, say $A_1^t, \dots, A_N^t$ such that for every $1 \leq m \leq N$, $(B_{\delta 2^{l-1}}(x))_{x \in A_m^t}$ is a family of disjoint balls and

$$K_t(l) \subset \bigcup_{m=1}^{N} \bigcup_{x \in A_m^t} B_{\delta 2^{l-1}}(x).$$

Therefore, from the previous estimate we get

$$\int_{K_t(l)} (v^h(x,t))^2 \,\mathrm{d}x \leq \frac{N(n)C(n,L)}{h} \int_\Omega \int_{u^h(x,t-h)}^{u^h(x,t)} \frac{\mathrm{sdist}_{u^h(\cdot,t-h)}}{h} \chi_{\{2^{l-1}h \leq \left|\mathrm{sdist}_{u^h(\cdot,t)}\right| \leq 2^{l+1}h\}} \,\mathrm{d}z \,\mathrm{d}x.$$

Summation over $l \leq l_0$ finally yields

$$\int_\Omega (v^h(x,t))^2 \,\mathrm{d}x \leq \frac{2N(n)C(n,L)}{h} \int_\Omega \int_{u^h(x,t-h)}^{u^h(x,t)} \frac{\mathrm{sdist}_{u^h(\cdot,t-h)}}{h} \,\mathrm{d}z \,\mathrm{d}x.$$

Recalling Lemma 3.2.1 we get

$$\int_\Omega (v^h)^2 \,\mathrm{d}x \leq \frac{C}{h} \int_\Omega \sqrt{1 + |\nabla u^h(\cdot, t-h)|^2} - \sqrt{1 + |\nabla u^h(\cdot, t)|^2} \,\mathrm{d}x.$$

Integrating in time from h to any $T = Mh > 0$ for some $M \in \mathbb{N}$ we get

$$\begin{aligned} \int_h^T \int_\Omega (v^h)^2 \,\mathrm{d}x &\leq C \left(\int_\Omega \sqrt{1 + |\nabla u^h(\cdot, 0)|^2} - \sqrt{1 + |\nabla u^h(\cdot, T)|^2} \,\mathrm{d}x \right) \\ &\leq C \int_\Omega \sqrt{1 + |\nabla u_0|^2} \,\mathrm{d}x. \end{aligned}$$

We can now pass into the limit $T \to +\infty$ to conclude the proof. □

Corollary 3.2.8. *For every $T > 0$ there exists a constant $C > 0$, independent of h, such that*

$$\int_h^T \int_\Omega |v^h(x,t)| \,\mathrm{d}x \,\mathrm{d}t \leq C.$$

Proof. Using Hölder's inequality and the previous lemma we get

$$\int_h^T \int_\Omega |v^h(x,t)| \,\mathrm{d}x \,\mathrm{d}t \leq (T|\Omega|)^{\frac{1}{2}} \left(\int_h^T \int_\Omega \left(v^h(x,t) \right)^2 \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{1}{2}} \leq C.$$

□

3.3. Convergence to Flat Flows

Let us now discuss in which sense we can pass to the limit $h \to 0$ in the sequence of approximate flows $(u^h)_{h>0}$.

First of all, we not that using the uniform control on the Lipschitz constant of our approximate flows, we can easily improve Proposition 3.2.6 to get equicontinuity in the L^∞-topology.

Corollary 3.3.1. *There exists a constant $C > 0$, independent of h, such that*

$$\left\| u^h(\cdot, t) - u^h(\cdot, s) \right\|_{L^\infty(\Omega)} \leq C|s - t|^{\frac{1}{2n+2}} \quad \forall s, t \geq h.$$

Proof. Indeed, let us fix some $s, t \geq h > 0$. Then, using the continuity of $u^h(\cdot, t)$ and $u^h(\cdot, s)$, we can assume, without loss of generality, that the supremum of the function $f(x) := \left|u^h(x, t) - u^h(x, s)\right|$ is positive and attained at some point $x_0 \in \Omega$. As f is a $2L$-Lipschitz function, it has to stay above the function l given by

$$l(x) := \max\{0, \|f\|_{L^\infty(\Omega)} - 2L|x - x_0|\}.$$

Consequently, as f is vanishing on $\partial\Omega$, by a simple computation we see that

$$\|f\|_{L^1(\Omega)} \geq \|l\|_{L^1(\Omega)} = C\|f\|^{n+1}_{L^\infty(\Omega)},$$

where $C > 0$ is a constant depending only on n and L. Combining this estimate with Proposition 3.2.6 gives the desired result. ☐

Corollary 3.3.2. *Let $(u^{h_k})_{k\in\mathbb{N}}$ be a sequence of approximate flows with $h_k \to 0$. Then there exists some $u \in C(\overline{\Omega} \times [0, +\infty[)$ such that up to possibly passing to a (not relabeled) subsequence, for every $0 < T < +\infty$ we have*

$$u^{h_k} \to u \quad \text{in } L^\infty(\Omega \times [0, T]) \quad (k \to +\infty).$$

Moreover, we have

$$\|u(\cdot, t) - u(\cdot, s)\|_{L^\infty(\Omega)} \leq C|s - t|^{\frac{1}{2n+2}}. \tag{3.12}$$

In particular, $u \in C^{0,\kappa}(\Omega \times [0, +\infty[)$, for $\kappa := \frac{1}{2n+2}$ and for every $t > 0$, $u(\cdot, t) \in \mathrm{Lip}(\Omega)$ with $\mathrm{Lip}(u(\cdot, t)) \leq L$, where L is the constant from Theorem 3.1.3.

Definition 3.3.1. Using the same notation as [3], any u as in the above corollary shall be called a *flat flow.*

Proof. We argue in two steps. First of all, we establish convergence on each time slice and then we show that we have convergence in spacetime. Let us start from the simple observation that for every $t > 0$ the set

$$\left\{u^{h_k}(\cdot, t) \in \mathrm{Lip}(\Omega) : \, k \in \mathbb{N}\right\}$$

is uniformly bounded and (due to the uniform Lipschitz bound) also uniformly equicontinuous. Hence, by Arzela-Ascoli, there exists a subsequence $(h_{k'})_{k\in\mathbb{N}}$ and some L-Lipschitz function v_t such that $u^{h_{k'}}(\cdot, t) \to v_t$ uniformly. By choosing a diagonal sequence we can easily assume that – up to passing to a subsequence – $(u^{h_k}(\cdot, t))_{k\in\mathbb{N}}$ is converging uniformly for every $t \in \mathbb{Q} \cap [0, +\infty[$. To see that $(u^{h_k}(\cdot, t))_{k\in\mathbb{N}}$ converges uniformly for *all* $t > 0$ it suffices to recall Corollary 3.3.1. Indeed, let us consider some $t \in [0, +\infty[\setminus\mathbb{Q}$ and fix $\varepsilon > 0$ arbitrary. By density, we find some $t_0 \in \mathbb{Q} \cap [0, +\infty[$ such that $C|t-t_0|^{\frac{1}{2n+2}} \leq \varepsilon$, where C is the constant from Corollary 3.3.1. Hence, if we take the supremum in Ω on both sides of the inequality

$$\Big|u^{h_n}(\cdot,t) - u^{h_m}(\cdot,t)\Big| \leq \\ \Big|u^{h_n}(\cdot,t) - u^{h_n}(\cdot,t_0)\Big| + \Big|u^{h_n}(\cdot,t_0) - u^{h_m}(\cdot,t_0)\Big| + \Big|u^{h_m}(\cdot,t_0) - u^{h_m}(\cdot,t)\Big|,$$

we get, due to the choice of t_0, that

$$\Big\|u^{h_n}(\cdot,t) - u^{h_m}(\cdot,t)\Big\|_{L^\infty(\Omega)} \leq 2\varepsilon + \Big\|u^{h_n}(\cdot,t_0) - u^{h_m}(\cdot,t_0)\Big\|_{L^\infty(\Omega)}.$$

Since $(u^{h_k}(\cdot,t_0))_{k\in\mathbb{N}}$ converges uniformly, for $n,m\in\mathbb{N}$, large enough, the remaining term can also be made smaller than ε. This shows that $(u^{h_k}(\cdot,t))_{k\in\mathbb{N}}$ is Cauchy and hence converges in $L^\infty(\Omega)$ as well. This fully justifies the following definition:

$$u(x,t) := \lim_{k\to\infty} u^{h_k}(x,t).$$

In particular, the uniform convergence of $(u^{h_k}(\cdot,t))_{k\in\mathbb{N}}$ to u allows us to deduce that $u(\cdot,t)$ is again Lipschitz with Lipschitz constant less or equal to L. Continuity in space is a simple consequence of the fact that u^{h_k} are continuous in space and the convergence to u is uniform. The continuity in time is a simple consequence of Corollary 3.3.1. Moreover, (3.12), and thus the Hölder-continuity of u, follow simply from passing into the limit in the estimate in Corollary 3.3.1.

In the second step, we fix now $T>0$ and consider the maps $f_k, f : [0,T] \to L^\infty(\Omega)$ defined via

$$f_k(t) := u^{h_k}(\cdot,t), \quad f(t) := u(\cdot,t),$$

Rephrasing the first step, we showed that $f_k \to f$ point-wise (in the L^∞-topology). Hence, as usual, the uniform equicontinuity of the maps f_k (Corollary 3.3.1) improves this point-wise convergence to a uniform convergence (for details, see Proposition B.2). It remains to observe that the uniform convergence of f_k to f is equivalent to the convergence of u^{h_k} to u in $L^\infty(\Omega\times[0,T])$.

□

3.4. Existence of Distributional Solutions

We fix a subsequence $(h_k)_{k\in\mathbb{N}}$ and $u : \Omega\times[0,+\infty[$ as in Corollary 3.3.2, such that for every $T>0$

$$u^{h_k} \to u \quad \text{in } L^\infty(\Omega\times[0,T]) \quad (k\to+\infty).$$

Often, we will use the abbreviation $\Omega_T = \Omega\times[0,T]$ which we hope, will not be confused with Ω_r as introduced in the basic notation at the beginning of this thesis.

We will now show, that using Proposition 3.2.7 and the regularity theory for the obstacle problem, we can improve the above convergence further.

Lemma 3.4.1. *Let $(u^{h_k})_{k\in\mathbb{N}}$ and u be as above. Then, for almost every time $t>0$, we have $u(\cdot,t)\in W^{2,2}(\Omega)$ with the estimates*

$$\|u^{h_k}(\cdot,t)\|_{W^{2,2}(\Omega)} \le C(1+\|v_{h_k}(\cdot,t)\|_{L^2(\Omega)}+\|D^2\psi\|_{L^2(\Omega)}), \tag{3.13}$$

$$\|u(\cdot,t)\|_{W^{2,2}(\Omega)} \le \liminf_{k\to\infty}\|u^{h_k}(\cdot,t)\|_{W^{2,2}(\Omega)}, \tag{3.14}$$

where the constant $C>0$ is independent of t. Moreover, $D^2u := D^2_x u \in L^2(\Omega\times[0,T[)$ for every $T<+\infty$.

Proof. Recall that for any $t>0$, $u^{h_k}(\cdot,t)$ solves the variational inequality:

$$\int_\Omega \frac{\nabla u^{h_k}(\cdot,t)\cdot\nabla(w-u^{h_k}(\cdot,t))}{\sqrt{1+|\nabla u^{h_k}(\cdot,t)|^2}} + v^{h_k}(\cdot,t)(w-u^{h_k}(\cdot,t))\mathrm{d}x \ge 0,$$

for every $w\in\mathrm{Lip}(\Omega)$ such that $w\ge\psi$ and $w_{|\partial\Omega}=u_{0|\partial\Omega}$. Hence we deduce by the standard $W^{2,2}$-estimates (see Proposition 2.3.1) that $u^{h_k}(\cdot,t)$ belongs to $W^{2,2}(\Omega)$ and satisfies the estimate (3.13). Moreover, by Proposition 3.2.7 we can even deduce that $D^2u\in L^2(\Omega\times[0,T[)$ for finite T. Using Fatou's lemma and again Proposition 3.2.7 we have for every $T>h>0$:

$$0\le\int_h^T\left(\liminf_{k\to\infty}\int_\Omega\left(v^{h_k}\right)^2\mathrm{d}x\right)\mathrm{d}t \le \liminf_{k\to\infty}\int_h^T\int_\Omega\left(v^{h_k}\right)^2\mathrm{d}x\,\mathrm{d}t\le C.$$

Consequently, for almost every $t\in[h,T]$ we find a subsequence $\Lambda_t\subset\mathbb{N}$ such that

$$\sup_{k\in\Lambda_t}\|v^{h_k}(\cdot,t)\|_{L^2(\Omega)}<+\infty,$$

which, using (3.13), results in

$$\sup_{k\in\Lambda_t}\|u^{h_k}(\cdot,t)\|_{W^{2,2}(\Omega)}<+\infty.$$

Since T and h can be chosen arbitrarily large and small respectively, we can indeed find such a subsequence Λ_t for almost every $t\in[0,+\infty[$. By the Rellich-Kondrachov compactness theorem, given Λ_t, we can extract a subsequence $\Lambda_t'\subset\mathbb{N}$ such that $(u^{h_k}(\cdot,t))_{k\in\Lambda_t'}$ converges in $W^{1,2}(\Omega)$ to some $v\in W^{2,2}(\Omega)$. However, since all time-slices (of the full sequence) converge in $L^\infty(\Omega)$ to $u(\cdot,t)$ we can identify $v=u(\cdot,t)$. In order to derive (3.14) we choose a subsequence $\tilde\Lambda_t$ such that

$$\lim_{k\in\tilde\Lambda_t,\,k\to\infty}\|u^{h_k}(\cdot,t)\|_{W^{2,2}(\Omega)}=\liminf_{k\to\infty}\|u^{h_k}(\cdot,t)\|_{W^{2,2}(\Omega)}.$$

Then, unless the limes inferior equals $+\infty$ (in which case the claim holds trivially), $(u^{h_k}(\cdot,t))_{k\in\tilde\Lambda_t}$ is bounded in $W^{2,2}(\Omega)$ and by using the same argument as before we can show that every subsequence of this sequence has a further subsequence converging to the same limit. Therefore we deduce that $u^{h_k}(\cdot,t) \to u(\cdot,t)$ in $W^{1,2}(\Omega)$ from which by lower-semi-continuity we derive the desired

$$\|u(\cdot,t)\|_{W^{2,2}(\Omega)} \leq \liminf_{k\in\tilde{\Lambda}_t,\, k\to\infty} \|u^{h_k}(\cdot,t)\|_{W^{2,2}(\Omega)} = \liminf_{k\to\infty} \|u^{h_k}(\cdot,t)\|_{W^{2,2}(\Omega)}.$$

□

Using an argument similar to one employed by Röger in [114, Lemma 4.2] we will next improve the uniform convergence in spacetime.

Lemma 3.4.2. *Let $(u^{h_k})_{k\in\mathbb{N}}$ and u be as in the previous lemma. Then for every $T > 0$ we have*

$$\nabla u^{h_k} \to \nabla u \quad \text{in } L^2(\Omega \times [0;T]) \quad (k\to\infty).$$

Proof. We introduce the following sequence of auxiliary functions: For $M > 0$ to be fixed momentarily, let

$$w_M^{h_k}(x,t) := \begin{cases} u^{h_k}(x,t), & \text{if } \int_\Omega \left(v^{h_k}(x,t)\right)^2 \leq M, \\ u(x,t), & \text{else.} \end{cases}$$

Let us also consider the following sets

$$A_{M,k} := \left\{ t\in[0,T] : \int_\Omega \left(v^{h_k}(x,t)\right)^2 > M \right\},$$
$$B_{M,k} := \left\{ t\in[0,T] : \int_\Omega \left(v^{h_k}(x,t)\right)^2 \leq M \right\}.$$

For any $T > 0$ we have

$$\|\nabla u^{h_k} - \nabla u\|_{L^2(\Omega_T)} \leq \|\nabla u^{h_k} - \nabla w_M^{h_k}\|_{L^2(\Omega_T)} + \|\nabla w_M^{h_k} - \nabla u\|_{L^2(\Omega_T)}. \tag{3.15}$$

We start by estimating the first term on the right hand side above and get

$$\|\nabla u^{h_k} - \nabla w_M^{h_k}\|^2_{L^2(\Omega_T)} = \int_{A_{k,M}} \int_\Omega |\nabla u^{h_k} - \nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \leq 4L^2|\Omega||A_{k,M}|.$$

Moreover, by Chebyshev:

$$|A_{M,k}| = |\{t\in[0,T] : \underbrace{\int_\Omega \left(v^{h_k}(x,t)\right)^2}_{=:f_k(t)} \geq M\}| \leq \frac{1}{M}\int_{\{f_k>M\}}\int_\Omega \left(v^{h_k}\right)^2 \leq \frac{C}{M},$$

where we used again Proposition 3.2.7 in the last inequality and where C is independent of k. Combining the last to inequalities we get

$$\|\nabla u^{h_k} - \nabla w_M^{h_k}\|_{L^2(\Omega_T)} \leq \frac{C}{M^{\frac{1}{2}}}, \tag{3.16}$$

uniformly in k. Before we can estimate the second term in (3.15) let us first prove the following assertion.

Claim: For every $t > 0$, $w_M^{h_k}(\cdot, t) \to u(\cdot, t)$ in $W^{1,2}(\Omega)$ as $(k \to \infty)$.

Proof of the claim: We fix some $t > 0$. By definition, $w_M^{h_k}(\cdot, t)$ is either equal to $u(\cdot, t)$ or – as seen in the proof of the previous lemma – it solves a variational inequality. However, in this latter case, the bound on the L^2-norm of the forcing term $v^{h_k}(\cdot, t)$ will lead to a uniform bound on the $W^{2,2}$-norm of $w_M^{h_k}(\cdot, t)$ (again via the regularity theory for variational inequalities cf. Proposition 2.3.1). By Rellich-Kondrachov we can thus extract a subsequence which converges in $W^{1,2}$ to $u(\cdot, t)$. Since arguing in this way, we can extract a converging subsequence (having the same limit, namely $u(\cdot, t)$) from *any* initial subsequence of $(w_M^{h_k}(\cdot, t))_{k\in\mathbb{N}}$, we proved that $(w_M^{h_k}(\cdot, t))_{k\in\mathbb{N}}$ converges in $W^{1,2}$ along the full sequence.

Now we would like to show that using this convergence, the second term in (3.15) vanishes as $k \to \infty$. Indeed, it suffices to show that

$$f_k(t) := \int_\Omega \left| \nabla w_M^{h_k}(\cdot, t) - \nabla u(\cdot, t) \right|^2 \mathrm{d}x$$

converges to zero in $L^1([0, T])$. This follows for instance from Lebesgue's dominated convergence theorem since the convergence of $(w_M^{h_k}(\cdot, t))_{k\mathbb{N}}$ in $W^{1,2}(\Omega)$ implies the pointwise convergence of f_k to zero and since $|f_k(t)|$ is bounded by $4L|\Omega|$ for every $k \in \mathbb{N}$. Finally for $\varepsilon > 0$ let M be large enough such that (cf. (3.16))

$$\left\| \nabla u^{h_k} - \nabla w_M^{h_k} \right\|_{L^2(\Omega_T)} \leq \frac{\varepsilon}{2}.$$

Then for $k \geq k_0(\varepsilon)$ we can also make sure that $\left(\int_0^T f_k \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{2}$. Thus, recalling (3.15), for k large enough we get $\left\| \nabla u^{h_k} - \nabla u \right\|_{L^2(\Omega_T)} \leq \varepsilon$ as desired. □

As a simple corollary we get the convergence of the area factors in L^2.

Lemma 3.4.3. *Let again $(u^{h_k})_{k\in\mathbb{N}}$ and u be as above. Then for every $T > 0$, we have*

$$\sqrt{1 + |\nabla u^{h_k}|^2} \to \sqrt{1 + |\nabla u|^2} \quad \text{in } L^2(\Omega_T) \quad \text{as } (k \to \infty).$$

Proof. Observing that

$$\begin{aligned} \left\| \sqrt{1 + |\nabla u^{h_k}|^2} - \sqrt{1 + |\nabla u|^2} \right\|_{L^2(\Omega_T)}^2 &= \int_0^T \int_\Omega \left| \int_{|\nabla u|}^{|\nabla u^{h_k}|} \frac{s}{\sqrt{1 + s^2}} \mathrm{d}s \right|^2 \mathrm{d}x \, \mathrm{d}t \\ &\leq \int_0^T \int_\Omega \left| |\nabla u| - |\nabla u^{h_k}| \right|^2 \mathrm{d}x \, \mathrm{d}t, \end{aligned}$$

the desired claim follows from the inverse triangle inequality and the previous lemma. □

Another consequence of Lemma 3.4.2 is the convergence of the mean curvature.

Proposition 3.4.4. *Let again $(u^{h_k})_{k\in\mathbb{N}}$ and u be as above. Then for every $T>0$, we have*

$$\operatorname{div}\left(\frac{\nabla u^{h_k}}{\sqrt{1+|\nabla u^{h_k}|^2}}\right) \rightharpoonup \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \quad \text{weakly in } L^2(\Omega_T),$$

as $(k\to\infty)$.

Proof. Step one: Fix $\phi \in C_c^\infty(\Omega\times[0,+\infty[)$. Integrating by parts and Hölder yields

$$\begin{aligned}
&\left|\int_0^T\int_\Omega \phi \operatorname{div}\left(\frac{\nabla u^{h_k}}{\sqrt{1+|\nabla u^{h_k}|^2}}\right)\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \phi \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\mathrm{d}x\,\mathrm{d}t\right| \\
&\le \int_0^T\int_\Omega \left|\nabla\phi\cdot\left(\frac{\nabla u^{h_k}}{\sqrt{1+|\nabla u^{h_k}|^2}} - \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right|\mathrm{d}x\,\mathrm{d}t \qquad (3.17) \\
&\le \|\nabla\phi\|_{L^2(\Omega_T)}\left\|\frac{\nabla u^{h_k}}{\sqrt{1+|\nabla u^{h_k}|^2}} - \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right\|_{L^2(\Omega_T)}.
\end{aligned}$$

Now observe that

$$\begin{aligned}
&\left|\frac{\nabla u^{h_k}}{\sqrt{1+|\nabla u^{h_k}|^2}} - \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right| \\
&\le \left|\frac{\nabla u^{h_k}-\nabla u}{\sqrt{1+|\nabla u^{h_k}|^2}}\right| + \left|\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right|\left|\frac{\sqrt{1+|\nabla u^{h_k}|^2}-\sqrt{1+|\nabla u|^2}}{\sqrt{1+|\nabla u^{h_k}|^2}}\right| \\
&\le \left|\nabla u^{h_k}-\nabla u\right| + \left|\sqrt{1+|\nabla u^{h_k}|^2}-\sqrt{1+|\nabla u|^2}\right|.
\end{aligned}$$

Hence, by Lemma 3.4.2 and Lemma 3.4.3 we see that the second factor in the last line of (3.17) converges to zero.

Step two: For arbitrary $\phi \in L^2(\Omega_T)$ we use a density argument. For $\varepsilon > 0$ let $\phi_\varepsilon \in C_c^\infty(\Omega\times[0,+\infty[)$ be such that $\|\phi_\varepsilon - \phi\|_{L^2(\Omega_T)} < \frac{\varepsilon}{K}$, for some $K>0$ to be fixed momentarily.

$$\begin{aligned}
&\left|\int_0^T\int_\Omega \phi\left(\operatorname{div}\left(\frac{\nabla u^{h_k}}{\sqrt{1+|\nabla u^{h_k}|^2}}\right) - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right)\mathrm{d}x\,\mathrm{d}t\right| \\
&\le \int_0^T\int_\Omega |\phi-\phi_\varepsilon|\left|\operatorname{div}\left(\frac{\nabla u^{h_k}}{\sqrt{1+|\nabla u^{h_k}|^2}}\right) - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right|\mathrm{d}x\,\mathrm{d}t \\
&+ \|\nabla\phi_\varepsilon\|_{L^2(\Omega_T)}\left\|\frac{\nabla u^{h_k}}{\sqrt{1+|\nabla u^{h_k}|^2}} - \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right\|_{L^2(\Omega_T)}.
\end{aligned}$$

As we saw in the first case, the second term on the right hand side converges to zero as $k \to +\infty$. In order to make the first term also small we can use the parameter K. Namely we estimate the first term by Hölder and choose

$$K = \max \left\{ \sup_{k\in\mathbb{N}} \left\| \left(\operatorname{div} \frac{\nabla u^{h_k}}{\sqrt{1+|\nabla u^{h_k}|^2}} \right) \right\|_{L^2(\Omega_T)}, \left\| \operatorname{div} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) \right\|_{L^2(\Omega_T)} \right\}.$$

Note that K is finite: By Proposition 2.3.4, the mean curvature of u^{h_k} can be bounded by

$$\left|v^{h_k}\right| + \left|\operatorname{div} \left(\frac{\nabla \psi}{\sqrt{1+|\nabla \psi|^2}} \right) \right| \in L^2(\Omega_T),$$

where we used Proposition 3.2.7 and the fact that $\psi \in C^{1,1}(\Omega)$. Moreover $D^2 u \in L^2(\Omega_T)$ as seen in Lemma 3.4.1. $\square$

Lemma 3.4.5. *Let $u, w \in C^1(\overline{\Omega})$ and $x_0 \in \Omega$ with $d := \operatorname{dist}_w(x_0, u(x_0)) < \operatorname{dist}_{\partial\Omega}(x_0)$. Then we have*

$$d\sqrt{1 + \inf_{B_d(x_0)} |\nabla w|^2} \le d|u(x_0) - w(x_0)| \le \sqrt{1 + \sup_{B_d(x_0)} |\nabla w|^2}. \tag{3.18}$$

Consequently, if $d = \operatorname{sdist}_w(x_0, u(x_0))$ we also have

$$\frac{u(x_0) - w(x_0)}{\sqrt{1+\sup_{B_d(x_0)} |\nabla w|^2}} \le \operatorname{sdist}_w(x_0, u(x_0)) \le \frac{u(x_0) - w(x_0)}{\sqrt{1+\inf_{B_d(x_0)} |\nabla w|^2}}, \tag{3.19}$$

and in case $d = -\operatorname{sdist}_w(x_0, u(x_0))$ we have

$$\frac{u(x_0) - w(x_0)}{\sqrt{1+\inf_{B_d(x_0)} |\nabla w|^2}} \le \operatorname{sdist}_w(x_0, u(x_0)) \le \frac{u(x_0) - w(x_0)}{\sqrt{1+\sup_{B_d(x_0)} |\nabla w|^2}}. \tag{3.20}$$

Proof. Since (3.18) obviously implies (3.19) and (3.20) we only have to prove this pair of bounds. The condition $d < \operatorname{dist}_{\partial\Omega}(x_0)$ guarantees that the distance of $(x_0, u(x_0))$ from the graph of w is attained at some point $(y_0, w(y_0))$ with $y_0 \in \Omega$. Moreover, it holds $|x_0 - y_0| \le r$. Without loss of generality, let us assume that $w(x_0) > u(x_0)$. From elementary computations we see that

$$w(y_0) - u(x_0) = \frac{d}{\sqrt{1+|\nabla w(y_0)|^2}} \quad \text{and} \quad |x_0 - y_0| = \frac{d|\nabla w(y_0)|}{\sqrt{1+|\nabla w(y_0)|^2}}.$$

Using these relations and the Lipschitzianity of w we can estimate further to get

$$\inf_{B_d(x_0)} |\nabla w||x_0 - y_0| \le w(x_0) - w(y_0) \le \sup_{B_d(x_0)} |\nabla w||x_0 - y_0|.$$

Combining these computations and using the monotonicity of $s \mapsto \frac{cs+1}{\sqrt{1+s^2}}$ in $[0, c]$ we finally deduce

$$w(x_0) - u(x_0) \leq d\,\frac{\sup_{B_d(x_0)} |\nabla w||\nabla w(y_0)| + 1}{\sqrt{1 + |\nabla w(y_0)|^2}} \leq d\sqrt{1 + \sup_{B_d(x_0)} |\nabla w|^2}.$$

The lower bound follows analogous and the estimates for $\mathrm{sdist} w(x_0, u(x_0))$ are then immediate. $\square$

Using the previous lemma and Proposition 3.2.7, we can now first of all show that u has also a weak partial derivative in time.

Corollary 3.4.6. *Let u be as in the beginning of this section. Then we have*

$$u_t \in L^2(\Omega \times [0, +\infty[).$$

Proof. It suffices to show that there exists a constant $C > 0$ such that for whenever $\phi \in C_c^1(\Omega \times]0, +\infty[)$ we have

$$\left| \int_{\Omega \times]0,+\infty[} u \partial_t \phi \right| \leq C \|\phi\|_{L^2(\Omega \times]0,+\infty[)}.$$

Let u^{h_k} be as in the beginning of this section. Due to the compact support of ϕ we get

$$\begin{aligned}\int_{\Omega \times]0,+\infty[} u \partial_t \phi &= \lim_{k\to+\infty} \int_\Omega \int_0^{+\infty} u^{h_k}(x,t) \frac{\phi(x, t + h_k) - \phi(x,t)}{h_k} \,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{k\to+\infty} \int_\Omega \int_0^{+\infty} \frac{u^{h_k}(x, t - h_k) - u^{h_k}(x,t)}{h_k} \phi(x,t) \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Next, we observe that since $u^{h_k} \to u$ in $L^\infty(\mathrm{supp}(\phi))$ as $k \to +\infty$, we can assure that for k large enough

$$\mathrm{dist}_{u^{h_k}(\cdot, t-h)}(x, u^{h_k}(x,t)) < \mathrm{dist}_{\partial\Omega}(x) \quad \forall (x,t) \in \mathrm{supp}(\phi).$$

Hence by Lemma 3.4.5 we know that for every $(x,t) \in \mathrm{supp}(\phi)$

$$\left| \frac{u^{h_k}(x,t) - u^{h_k}(x, t - h_k)}{h_k} \right| \leq C(L) \left| v^{h_k} \right| (x,t),$$

where v^{h_k} is the discrete velocity (cf. Definition 3.2.2). Consequently, we get

$$\left| \int_{\Omega \times]0,+\infty[} u \partial_t \phi \right| \leq C \limsup_{k\to+\infty} \int_{\Omega \times]0,+\infty[} \left| v^{h_k} \right| \phi.$$

The desired estimate now follows from Hölder's inequality and Proposition 3.2.7. $\square$

Let us now derive the analog of Proposition 2.2 in [96]. In order to simplify the notation we will henceforth write $(h \to 0)$, while in fact on should of course always assume that the convergence is only along a subsequence $(h_k)_{k\in\mathbb{N}}$.

Proposition 3.4.7. *Let v^h be the discrete velocity, as defined in Definition 3.2.2. Then, for every every $\phi \in C^1_c(\Omega \times [0,+\infty[)$ for $(h \to 0)$ we have*

$$\int_{2h}^{+\infty}\int_\Omega \left(\frac{1}{h}\left(u^h(x,t) - u^h(x,t-h)\right) - \sqrt{1+|\nabla u^h(x,t-h)|^2}v^h\right)\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t \to 0.$$

Proof. Fix ϕ as above and let h_0 be small enough and $T > 0$ large enough such that $\operatorname{supp}(\phi) \subset \Omega_{2\gamma\sqrt{h_0}} \times [0,T]$, where $\Omega_r = \{x \in \Omega \;:\; \operatorname{dist}(x,\partial\Omega) > r\}$. Without loss of generality, we assume henceforth that $h < h_0$. Let us next fix a time $t \in]2h,T[$. For any $x_0 \in \Omega_{\gamma_n\sqrt{h}}$ we let

$$A_{x_0} := \begin{cases} B_{\frac{1}{2}h^\alpha}(x_0), & \text{if } v^h(y,t) < h^{\alpha-1} \quad \forall y \in B_{\gamma_n\sqrt{h}}(x_0),\\ B_{\gamma_n\sqrt{h}}(x_0), & \text{else.}\end{cases}$$

As an application of Besicovitch's covering theorem there exists a finite collection of points $I \subset \Omega_{2\gamma\sqrt{h}}$ such that $(A_x)_{x\in I}$ covers $\Omega_{2\gamma\sqrt{h}}$.

$$\begin{aligned}&\left|\int_\Omega \left(\frac{1}{h}\left(u^h(x,t) - u^h(x,t-h)\right) - \sqrt{1+|\nabla u^h(x,t-h)|^2}v^h\right)\phi\,\mathrm{d}x\right|\\ &\qquad\le \sum_{x\in I}\int_{A_x}\left|\frac{1}{h}\left(u^h(x,t) - u^h(x,t-h)\right) - \sqrt{1+|\nabla u^h(x,t-h)|^2}v^h\right|\phi\,\mathrm{d}x.\end{aligned}$$

We will now estimate the contribution of each A_x separately, depending on whether it is a region of high curvature or not. Let $\frac{1}{2} < \alpha < 1$ be a constant to be fixed momentarily.

Case 1 (low curvature). Fix x_0 such that $A_{x_0} = B_{\frac{1}{2}h^\alpha}(x_0)$ i.e. such that we have $\operatorname{dist}_{u^h(\cdot,t-h)}(x,u^h(x,t)) < h^\alpha$ whenever $x \in A_{x_0}$. Consequently, by Corollary 3.2.8 we get

$$\begin{aligned}&\left|\frac{1}{h}\left(u^h(x,t) - u^h(x,t-h)\right) - \sqrt{1+|\nabla u^h(x,t-h)|^2}v^h\right|\\ &\le\left(\sqrt{1+\sup_{B_{h^\alpha}(x)}|\nabla u(\cdot,t-h)|^2} - \sqrt{1+\inf_{B_{h^\alpha}(x)}|\nabla u(\cdot,t-h)|^2}\right)\left|v^h(x,t)\right|\\ &=\int_{\inf_{B_{h^\alpha}(x)}|\nabla u(\cdot,t-h)|}^{\sup_{B_{h^\alpha}(x)}|\nabla u(\cdot,t-h)|}\frac{s}{\sqrt{1+s^2}}\,\mathrm{d}s\left|v^h(x,t)\right| \qquad (3.21)\\ &\le\left(\sup_{B_{h^\alpha}(x)}|\nabla u(\cdot,t-h)| - \inf_{B_{h^\alpha}(x)}|\nabla u(\cdot,t-h)|\right)\left|v^h(x,t)\right|\\ &\le\operatorname{osc}_{B_{h^\alpha}(x)}|\nabla u(\cdot,t-h)|\left|v^h(x,t)\right|.\end{aligned}$$

It remains to estimate this oscillation. Let us recall that by Proposition 2.3.4 for a.e. $x \in \Omega$ we have

$$f(x) := \operatorname{div}\left(\frac{\nabla u^h(x,t-h)}{\sqrt{1+|\nabla u(x,t-h)|^2}}\right) = \begin{cases} \operatorname{div}\left(\frac{\nabla\psi(x)}{\sqrt{1+|\nabla\psi(x)|^2}}\right) & \text{if } u^h(x,t-h)=\psi(x), \\ v^h(x,t-h) & \text{else.} \end{cases} \tag{3.22}$$

Consider now, for any $1 \le k \le n$, the function $w = \partial_{x_k} u^h(\cdot, t-h)$. It is easy to see, that $w \in W^{1,2}(\Omega)$ is a weak solution of

$$\operatorname{div}(A\nabla w) = \partial_k f,$$

for some uniformly elliptic matrix A. For details and properties of A we refer to the proof of Theorem A.3. Recall that from (3.22) and Proposition 3.2.4 we also know that

$$\|f\|_{L^\infty}(\Omega) \le Ch^{-\frac{1}{2}}.$$

We consider now the rescaling $w_h : B_1 := B_1(0;\mathbb{R}^n) \to \mathbb{R}$, $w_h(y) := \frac{1}{\sqrt{h}} w(\sqrt{h}y + x)$. It is straightforward to check that w_h is a weak solution of $v \in W^{1,2}(B_1)$, with

$$\operatorname{div}(A_h \nabla v) = \partial_k f_h,$$

where $A_h(x) = A(\sqrt{h}x)$ and $f_h(x) = f(\sqrt{h}x)$. Consequently, by De Giorgi-Nash (cf. Theorem 8.22 in [68]) we get that there are constants $0 < \beta < 1$ and $C > 0$, both depending only on n and L, such that

$$\operatorname{osc}_{B_r(x)}(w_h) \le Cr^\beta \left(\sup_{B_1} |w_h| + \|f_h\|_{L^\infty(B_1)}\right) \quad \forall r \le 1.$$

Noting that $|w| \le L$ and using the bound on f we can rewrite this estimate in terms of w to obtain

$$\operatorname{osc}_{B_{r\sqrt{h}}(x)}(w) \le Cr^\beta \left(L + \sqrt{h}(Ch^{-\frac{1}{2}})\right) \quad \forall r \le 1.$$

Finally we choose $r = h^\tau$ with $\tau = \alpha - \frac{1}{2}$ to end up with

$$\operatorname{osc}_{B_{h^\alpha}(x)}(w) \le Ch^{\tau\beta}.$$

Recalling the definition of w and (3.21) we have

$$\left|\frac{1}{h}\left(u^h(x,t) - u^h(x,t-h)\right) - \sqrt{1+|\nabla u^h|^2} v^h\right| \le Ch^{\tau\beta}\left|v^h(x,t)\right|. \tag{3.23}$$

Case 2 (high curvature) Since in this case, $A_x = B_{\gamma\sqrt{h}}(x)$, using Lemma 3.2.3 and Proposition 3.2.4 we easily estimate

$$\begin{aligned} &\int_{A_x} \left|\frac{1}{h}\left(u^h(x,t) - u^h(x,t-h)\right) - \sqrt{1+|\nabla u^h(x,t-h)|^2} v^h\right| |\phi| \,\mathrm{d}x \\ &\le C(\gamma\sqrt{h})^n \frac{1}{h}\left(C + \sqrt{1+L^2}\right) \left\|\operatorname{sdist}_{u^h(\cdot,t-h)}(x, u^h(x,t))\right\|_{L^\infty} \|\phi\|_{L^\infty} \\ &\le Ch^{\frac{n-1}{2}} \|\phi\|_{L^\infty}. \end{aligned} \tag{3.24}$$

Moreover, by assumption, we are now in the case that there exists a point $x_0 \in B_{\gamma\sqrt{h}}(x)$ such that $\left|\mathrm{sdist}_{u^h(\cdot,t-h)}(x_0, u^h(x_0,t))\right| \geq h^\alpha$. By Lemma 3.2.2, we know that

$$\mathrm{graph}(u^h(\cdot,t)_{|B_{\delta\frac{h^\alpha}{2}}}) \subset B_{\frac{h^\alpha}{2}}((x_0, u(x_0)); \mathbb{R}^{n+1}),$$

which by the triangle inequality implies

$$\left|v^h(y,t)\right| = \frac{\left|\mathrm{sdist}_{u^h(\cdot,t-h)}(y, u^h(y,t))\right|}{h} \geq \frac{h^{\alpha-1}}{2} \quad \forall y \in B_{\delta\frac{h^\alpha}{2}}(x_0).$$

Since $B_{\delta\frac{h^\alpha}{2}}(x_0) \subset B_{2\gamma\sqrt{h}}(x)$ we can estimate

$$Ch^{\alpha(n+1)-2} \leq \int_{B_{\delta\frac{h^\alpha}{2}}(x_0)} \left|v^h(x,t)\right|^2 \mathrm{d}x \leq \int_{B_{2\gamma\sqrt{h}}(x)} \left|v^h(x,t)\right|^2 \mathrm{d}x. \tag{3.25}$$

Combining (3.24) and (3.25) we get

$$\begin{aligned}\int_{A_x} \left|\frac{1}{h}\Big(u^h(x,t) - u^h(x,t-h)\Big) - \sqrt{1+|\nabla u^h|^2}v^h\right| |\phi| \,\mathrm{d}x& \\ \leq Ch^{\frac{n-1}{2}-\alpha(n+2)+2} \int_{B_{2\gamma\sqrt{h}}(x)} \left|v^h(x,t)\right| \mathrm{d}x.& \quad (3.26)\end{aligned}$$

Summing now over all $x \in A$, thanks to (3.23) and (3.26) we obtain

$$\begin{aligned}\int_{\Omega} |\phi| \left|\frac{1}{h}\Big(u^h(x,t) - u^h(x,t-h)\Big) - \sqrt{1+|\nabla u^h|^2}v^h\right| |\phi| \,\mathrm{d}x& \\ \leq Ch^{\tau\beta} \int_\Omega \left|v^h(x,t)\right| \mathrm{d}x + h^{\frac{n+3}{2}-\alpha(n+2)} \int_\Omega \left|v^h(x,t)\right|^2 \mathrm{d}x.&\end{aligned}$$

Since the constant C on the right hand side is independent of t we can now we integrate in time and use Corollary 3.2.8 to get

$$\begin{aligned}&\int_{2h}^T \int_\Omega \left|\frac{1}{h}\Big(u^h(x,t) - u^h(x,t-h)\Big) - \sqrt{1+|\nabla u^h|^2}v^h\right| |\phi| \,\mathrm{d}x \,\mathrm{d}t \\ &\leq C(h^{\tau\beta} \int_{2h}^T \int_\Omega \left|v^h(x,t)\right| \mathrm{d}x \,\mathrm{d}t + h^{\frac{n+1}{2}-\alpha(n+1)+1}) \int_{2h}^T \int_\Omega \left|v^h(x,t)\right|^2 \mathrm{d}x \,\mathrm{d}t \\ &\leq C(h^{\tau\beta} + h^{\frac{n+3}{2}-\alpha(n+2)}).\end{aligned}$$

Letting $\alpha \in]\frac{1}{2}, \frac{n+3}{2(n+2)}[$, the proposition follows by taking the limit $h \to 0$. $\square$

Finally, we are ready to prove the existence of distributional solutions.

Theorem 3.4.8. *There exists $u \in C^{0,\kappa}(\Omega \times [0,+\infty[)$, where $\kappa = \frac{1}{2n+2}$ satisfying*

i) $u(\cdot, 0) = u_0$,

ii) $u(x,t) = u_0(x)$ *for all* $x \in \partial\Omega$, *and every* $t \geq 0$,

iii) $u(\cdot,t) \geq \psi$ *for every* $t > 0$,

iv) $D_x^2 u,\ u_t \in L^2(\Omega \times [0,T])$ *for all* $T > 0$,

v) *For every* $\phi \in C_c^1(\Omega \times [0,+\infty[)$ *with* $\phi \geq 0$ *on* $\{(x,t) \in \Omega \times [0,+\infty[\colon u(x,t) = \psi(x)\}$ *we have*

$$\int_0^{+\infty} \int_\Omega u_t \phi \,\mathrm{d}x\,\mathrm{d}t \geq \int_0^{+\infty} \int_\Omega \sqrt{1+|\nabla u|^2} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \phi \,\mathrm{d}x\,\mathrm{d}t. \tag{3.27}$$

Remark 3.4.1. Since $\{(x,t) \in \Omega \times [0,+\infty[\colon u(x,t) > \psi(x)\}$ is open, localizing in (3.27) yields that almost everywhere we have

$$\begin{cases} u_t = \sqrt{1+|\nabla u|^2} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) & \text{on } \{(x,t) \colon u(x,t) > \psi(x)\}, \\ u_t \geq \max\left\{0, \sqrt{1+|\nabla u|^2} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right\} & \text{on } \{(x,t) \colon u(x,t) = \psi(x)\}. \end{cases}$$

Note the discrepancy with the heuristics in the introduction where equality in the second case was expected.

Proof (of Theorem 3.4.8). Let $T > h > 0$ be such that $\operatorname{supp}(\phi) \subset \Omega \times [0, T-h]$

$$\begin{aligned} \int_{2h}^T \int_\Omega \left(u^h - u^h(\cdot,\cdot-h)\right)\phi &= \int_{2h}^T \int_\Omega u^h \phi - \int_h^{T-h} \int_\Omega u^h \phi(\cdot,\cdot+h) \\ &= \int_{2h}^T \int_\Omega u^h \left(\phi - \phi(\cdot,\cdot+h)\right) - \int_h^{2h} \int_\Omega u^h \phi(\cdot,\cdot+h). \end{aligned} \tag{3.28}$$

We would like to divide both sides by h and pass into the limit. Recalling that $u^h \to u$ in $L^\infty(\Omega \times [0,T])$ and since

$$\chi_{[2h,T]} \frac{\phi - \phi(\cdot,\cdot+h)}{h} \to -\phi_t \quad \textit{in } L^1(\Omega \times [0,T]) \quad \textit{as } h \to 0,$$

we get that

$$\lim_{h\to 0} \frac{1}{h} \int_{2h}^T \int_\Omega u^h \left(\phi - \phi(\cdot,\cdot+h)\right) = -\int_0^T \int_\Omega u \phi_t = \int_0^T \int_\Omega u_t \phi. \tag{3.29}$$

For the second term, by continuity of: $[h,2h] \ni t \mapsto \int_\Omega u^h(x,t)\phi(x,t+h)\mathrm{d}x$, the mean value theorem implies that

$$\lim_{h\to 0} \frac{1}{h} \int_h^{2h} \int_\Omega u^h \phi(x,t+h) \mathrm{d}x\,\mathrm{d}t = \int_\Omega u_0(x)\phi(x,0)\mathrm{d}x. \tag{3.30}$$

Therefore, by Proposition 3.4.7, (3.28), (3.29) and (3.30) we obtain

$$\int_0^T \int_\Omega u_t \phi - \int_\Omega u_0 \phi(\cdot, 0) = \lim_{h \to 0} \int_{2h}^T \int_\Omega \sqrt{1 + |\nabla u^h(x, t-h)|^2} v^h \phi \, \mathrm{d}x \, \mathrm{d}t. \tag{3.31}$$

In order to analyze the right hand side of this equation we introduce the following sets

$$\begin{aligned} A_1^h &:= \{(x,t) \in \Omega \times]2h, T[: \psi(x) < u^h(x,t)\}, \\ A_2^h &:= \{(x,t) \in \Omega \times]2h, T[: \psi(x) = u^h(x,t) = u(x,t)\}, \\ A_3^h &:= \{(x,t) \in \Omega \times]2h, T[: \psi(x) = u^h(x,t) < u(x,t)\}. \end{aligned}$$

First of all we note that for every $h > 0$ these three sets constitute a disjoint partition of $\Omega \times]2h, T[$. Moreover, by (2.24) in Proposition 2.3.4 we know that on A_1^h we have

$$v^h = \operatorname{div}\left(\frac{\nabla u^h}{\sqrt{1 + |\nabla u^h|^2}}\right),$$

while on A_2^h due to (2.25) we only know

$$v^h \geq \operatorname{div}\left(\frac{\nabla u^h}{\sqrt{1 + |\nabla u^h|^2}}\right).$$

However, since on A_2^h, by assumption $\phi \geq 0$ we can deduce nevertheless that for all $h > 0$

$$\begin{aligned} &\int_{2h}^T \int_\Omega \sqrt{1 + |\nabla u^h(x,t-h)|^2} v^h \phi \, \mathrm{d}x \, \mathrm{d}t \geq \\ &\int_{A_1^h \cup A_2^h} \sqrt{1 + |\nabla u^h(x,t-h)|^2} \operatorname{div}\left(\frac{\nabla u^h}{\sqrt{1 + |\nabla u^h|^2}}\right) \phi \, \mathrm{d}x \, \mathrm{d}t + F(h), \end{aligned} \tag{3.32}$$

where

$$F(h) := \int_{A_3^h} \sqrt{1 + |\nabla u^h(x,t-h)|^2} v^h \phi \, \mathrm{d}x \, \mathrm{d}t.$$

As an auxiliary result, we will now prove that $|A_3^h| \to 0$ as $h \to 0$. We start by selecting a subsequence $(h_k)_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$

$$u^{h_k}(x,t) > u(x,t) - \frac{1}{k} \quad \forall (x,t) \in \Omega \times]0, T[.$$

By contradiction, we assume now that there exists $\varepsilon_0 > 0$ and a further subsequence $\Lambda \subset \mathbb{N}$ such that

$$\left|A_3^{h_k}\right| \geq \varepsilon_0 \quad \forall k \in \Lambda.$$

In addition, for every $k \in \mathbb{N}$ we have

$$A_3^{h_k} \subset \{(x,t) \in \Omega \times]0,T[:\ 0 < u(x,t) - \psi(x) < \frac{1}{k}\} =: E_k,$$

and hence, by monotonicity of the measure, $|E_k| \geq \varepsilon_0$ for all $k \in \Lambda$. Finally, since $E_{k+1} \subset E_k$ for every k we have

$$\left| \bigcap_{k \in \Lambda} E_k \right| = \lim_{\substack{k \to +\infty \\ k \in \Lambda}} |E_k| \geq \varepsilon_0.$$

The contradiction follows now from observing that $\bigcap_{k \in \Lambda} E_k = \emptyset$.

A first consequence of this auxiliary result is now that

$$F(h) \to 0, \text{ as } h \to 0. \tag{3.33}$$

Indeed, by Proposition 3.2.7 we know that

$$\left\| \sqrt{1 + |\nabla u^h(\cdot, \cdot - h)|^2} v^h \phi \right\|_{L^2(\Omega \times [0,T])} \leq C(L,\phi) \left\| v^h \right\|_{L^2(\Omega \times [0,T])} \leq C.$$

Thus, by Hölder's inequality

$$|F(h)| \leq \left| A_3^h \right|^{\frac{1}{2}} C,$$

and (3.33) follows. As a second consequence of the convergence of $\left| A_3^h \right|$ we can now also pass into the limit in the first term of the right hand side of (3.32) which we can rewrite, using characteristic functions, as follows

$$\int_{A_1^h \cup A_2^h} G_h \phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \chi_{A_1^h \cup A_2^h} G_h \phi \,\mathrm{d}x\,\mathrm{d}t,$$

where for $(x,t) \in \Omega \times]0,T[$ we set

$$G_h(x,t) := \sqrt{1 + |\nabla u^h(x,t-h)|^2} \operatorname{div} \left(\frac{\nabla u^h}{\sqrt{1 + |\nabla u^h|^2}} \right).$$

Here and in the following we adopt the following convention: Whenever $t < 0$, we set $\nabla u^h(x,t) := \nabla u(x,t) := \nabla u(x,0)$ for every $x \in \Omega$. We claim that $\chi_{A_1^h \cup A_2^h}$ converges to $\chi_{\Omega \times]0,T[}$ in $L^2(\Omega \times]0,T[)$. Indeed, we have

$$\left\| \chi_{A_1^h \cup A_2^h} - 1 \right\|^2_{L^2(\Omega \times]0,T[} = \int_0^T \int_\Omega \left| \chi_{A_3^h} \right|^2 = \left| A_3^h \right| \to 0 \quad (h \to 0).$$

Our next claim is that G_h converges weakly in $L^2(\Omega \times]0,T[)$. We will argue in three steps. First of all, we want to show that

$$\sqrt{1 + |\nabla u^h(x,t-h)|^2} \to \sqrt{1 + |\nabla u(x,t)|^2} \quad \text{in } L^2(\Omega \times]0,T[). \tag{3.34}$$

To deduce this, analogous to the proof of Lemma 3.4.3, if suffices to show that

$$\nabla u^h(x,t-h) \to \nabla u(x,t) \quad \text{in } L^2(\Omega\times]0,T[) \quad \text{as } (h \to 0).$$

For almost every $(x,t) \in \Omega\times]0,T[$ we have

$$\begin{aligned}&\Big|\nabla u^h(x,t-h) - \nabla u(x,t)\Big|^2 \\ &\qquad \le 2\Big|\nabla u^h(x,t-h) - \nabla u(x,t-h)\Big|^2 + 2|\nabla u(x,t-h) - \nabla u(x,t)|^2,\end{aligned}$$

and the claim follows by integrating this inequality in space and time. Indeed, on the one hand side, by a simple change of variables and Lemma 3.4.3 we get

$$\begin{aligned}\int_0^T \int_\Omega \Big|\nabla u^h(x,t-h) - \nabla u(x,t-h)\Big|^2 &= \int_h^{T-h} \int_\Omega \Big|\nabla u^h(x,s) - \nabla u(x,s)\Big|^2 \\ &\le \int_0^T \int_\Omega \Big|\nabla u^h(x,s) - \nabla u(x,s)\Big|^2 \to 0.\end{aligned}$$

On the other hand, since $\nabla u \in L^2(\Omega\times]0,T[)$ we can also show that

$$\nabla u(x,t-h) \to \nabla u(x,t) \quad \text{in } L^2(\Omega\times]0,T[) \text{ as } (h \to 0).$$

Indeed, since $\nabla u(x,t-h) \to \nabla u(x,t)$ pointwise, the desired convergence follows by dominated convergence as $|\nabla u(x,t-h) - \nabla u(x,t)| \le 2L$. Thus we proved (3.34). Secondly, we recall that by Proposition 3.4.4, as $h \to 0$,

$$\operatorname{div}\left(\frac{\nabla u^h}{\sqrt{1+|\nabla u^h|^2}}\right) \rightharpoonup \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \quad \text{weakly in } L^2(\Omega\times[0,T]). \tag{3.35}$$

Therefore, combing (3.34) and (3.35), by *weak-strong*-convergence we get that the product of the two terms, i.e. G_h satisfies

$$G_h \rightharpoonup \sqrt{1+|\nabla u|^2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \quad \text{weakly in } L^1(\Omega\times]0,T[).$$

In a third step, we can now improve this convergence to a weak-$L^2(\Omega\times]0,T[)$ convergence. Indeed, we note that G_h is the product of a term that is bounded in $L^\infty(\Omega\times]0,T[)$ and a term which is weakly convergent in $L^2(\Omega\times]0,T[)$. Therefore, we deduce that G_h is bounded in $L^2(\Omega\times]0,T[)$ and thus weakly convergent in $L^2(\Omega\times]0,T[)$ (cf. [45, Theorem 1.9.3]). Since weak limits are unique, we get the desired

$$G_h \rightharpoonup \sqrt{1+|\nabla u|^2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \quad \text{weakly in } L^2(\Omega\times]0,T[).$$

Recalling that $\chi_{A_1^h \cup A_2^h}$ converges to 1 in $L^2(\Omega\times]0,T[)$ we can use once again *weak-strong*-convergence to deduce that the product of $\chi_{A_1^h \cup A_2^h}$ and G_h converges weakly in $L^1(\Omega\times]0,T[)$ so that we finally get

$$\lim_{h\to 0} \int_{A_1^h \cup A_2^h} G_h \phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^{+\infty} \int_\Omega \sqrt{1+|\nabla u|^2} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\phi. \tag{3.36}$$

Combining (3.31), (3.32), (3.33) and (3.36) we deduce the corollary. □

3.5. Asymptotic Limit $t \to +\infty$

Proposition 3.5.1. *Let u be a distributional solution as in Theorem 3.4.8 and let v be the minimal surface above ψ which coincides with u_0 on the boundary as in Proposition 2.5.1. Then we have $u(\cdot,t) \to v$ uniformly in Ω as $t \to +\infty$.*

In the proof of this proposition we make us of the following Chebyshev-type lemma.

Lemma 3.5.2. *Let $f \in L^1(]0,+\infty[)$. Then, for every $k \in \mathbb{N}$, there exists $T_k > 0$ such that*

$$\left|\left\{t \in]0,+\infty[:\ |f(t)| > \frac{1}{k}\right\} \cap [T_k,+\infty[\right| \le \frac{1}{k}.$$

Proof. Assume by contradiction that there exists some $k_0 \in \mathbb{N}$ such that for every $T > 0$:

$$\left|\left\{t \in]0,+\infty[:\ |f(t)| > \frac{1}{k_0}\right\} \cap [T,+\infty[\right| > \frac{1}{k_0}.$$

Then, for every $T > 0$, we would have

$$\int_T^{+\infty} |f| \,\mathrm{d}x \ge \int_T^{+\infty} \frac{1}{k_0}\chi_{\{|f|>\frac{1}{k_0}\}} \,\mathrm{d}x \ge \frac{1}{k_0^2}.$$

This contradicts the fact that, since f belongs to $L^1(]0,+\infty[)$, the integral on the left hand side converges to zero as $T \to +\infty$. □

Proof of Proposition 3.5.1. We start by recalling that, according to Corollary 3.4.6, u_t belongs to $L^2(\Omega\times]0,+\infty[)$. Hence, by Fubini, $f_1(t) := \int_\Omega |u_t(x,t)|^2 \,\mathrm{d}x$ belongs to $L^1(]0,+\infty[)$. Additionally, we recall that by Fatou's lemma and Proposition 3.2.7

$$f_2(t) := \liminf_{k\to+\infty} ||v^{h_k}(\cdot,t)||^2_{L^2(\Omega)},$$

also belongs to $L^1(]0,+\infty[)$, where v^{h_k} is defined as in Definition 3.2.2 and where we assume that $(u^{h_k})_{k\in\mathbb{N}}$ is a sequence of approximate flows with $u^{h_k} \to u$ uniformly as $k \to +\infty$. Applying Lemma 3.5.2 to $f := f_1 + f_2$ we see that for every $k \in \mathbb{N}$ the set

$\{t \in [T_k, +\infty[\colon |f| \le \frac{1}{k}\}$ has infinite measure (in fact, positive measure would also be sufficient at this point). Noting that f_1 belongs to $L^1(]0, +\infty[)$ and

$$h(t) := \int_\Omega \sqrt{1 + |\nabla u|^2} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \mathrm{d}x,$$

belongs to $L^1_{\mathrm{loc}}(]0, +\infty[)$, we see that almost every $t > 0$ is simultaneously a Lebesgue-point for f_1 and h. Therefore, using also Lemma 3.4.1, we can easily construct an increasing sequence $t_k \to \infty$ as $(k \to \infty)$ such that

(a) t_k is a Lebesgue-point for f_1 and h,

(b) $\int_\Omega |u_t(x, t_k)|^2 \, \mathrm{d}x \le \frac{1}{k}$,

(c) $\|u(\cdot, t_k)\|_{W^{2,2}(\Omega)} \le C$, where $C > 0$ is not depending on k.

Owing to (c) and the uniform Lipschitz bound (in space), up to possibly passing to a subsequence, we can assume that $(u(\cdot, t_k))_{k \in \mathbb{N}}$ converges (strongly) in $L^\infty(\Omega)$ and $W^{1,2}(\Omega)$ and weakly in $W^{2,2}(\Omega)$ to some element $v \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Morally, we would now like to use (3.27) with tests of the form $\Phi(x,t) = \phi(x)\eta(t)$, where $\phi \in C^1_c(\Omega)$ with $\phi \ge 0$ on $\{v = \psi\}$ and $0 \le \eta \in C^1_c(]0, +\infty[)$. We could then use η to localize at times t_k and pass into the limit. However, in general there is no reason why such Φ should by an admissible test, i.e. why Φ should satisfy $\Phi \ge 0$ on $\{(x,t) \,:\, u(x,t) = \psi(x)\}$. Therefore, we need to argue more carefully. Namely, we will show that for every ϕ and $(t_k)_{k \in \mathbb{N}}$ as above, there exists $\Phi \in C^1(\Omega \times]0, +\infty[)$ with the following three properties:

(i) $\Phi \ge 0$ on $\{(x,t) \,:\, u(x,t) = \psi(x)\}$,

(ii) $\Phi(\cdot, t) \in C^1_c(\Omega) \quad \forall t > 0$,

(iii) $\Phi(\cdot, t_k) \to \phi$ uniformly as $(k \to +\infty)$.

Postponing the construction of Φ and assuming its existence, we note that for any $0 \le \eta \in C^1_c(]0, +\infty[)$ we get

$$\widetilde{\Phi} := \Phi\,\eta \in C^1_c(\Omega \times]0, +\infty[) \quad \text{and} \quad \widetilde{\Phi} \ge 0 \text{ on } \Lambda_u = \{(x,t) \,:\, u(x,t) = \psi(x)\}.$$

Hence, we can test (3.27) with $\widetilde{\Phi}$ to get

$$\int_0^{+\infty} \left(\int_\Omega u_t \Phi \, \mathrm{d}x \right) \eta \, \mathrm{d}t \ge \int_0^{+\infty} \left(\int_\Omega \sqrt{1 + |\nabla u|^2} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \Phi \, \mathrm{d}x \right) \eta \, \mathrm{d}t.$$

Recalling property (a) and taking $\eta = \eta_l$, a sequence with $\eta_l \overset{(l \to \infty)}{\to} \delta_{t_k}$ (the Dirac-δ at t_k), we obtain that for every $k \in \mathbb{N}$

$$\int_\Omega u_t(x, t_k)\Phi(x, t_k) \, \mathrm{d}x \ge \int_\Omega \sqrt{1 + |\nabla u(x, t_k)|^2} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)(x, t_k) \, \Phi(x, t_k) \, \mathrm{d}x.$$

By property (b), $u_t(\cdot, t_k) \to 0$ in $L^2(\Omega)$ as $(k \to +\infty)$. Moreover, very similar to the arguments which led to (3.35), by weak-strong convergence we see that the product of the square-root term and the divergence term converges weakly in $L^1(\Omega)$ so that together with property (iii) we can pass into the limit $k \to +\infty$ to obtain that

$$0 \geq \int_\Omega \sqrt{1+|\nabla v|^2}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)\phi \, \mathrm{d}x, \tag{3.37}$$

for every $\phi \in C_c^1(\Omega)$ with $\phi \geq 0$ on $\{x \in \Omega \ : \ v = \psi\}$. We claim that this inequality immediately implies

$$0 \leq \int_\Omega \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla\phi(x) \, \mathrm{d}x. \tag{3.38}$$

To see this, we start by considering $l : \mathbb{R}^n \to \mathbb{R}$ defined as $l(x) := (1+|\nabla v(x)|^2)^{-\frac{1}{2}}$ for $x \in \Omega$ and $l(x) = 0$ otherwise. Let $l_\varepsilon := (l * \rho_\varepsilon)_{|\Omega}$ be as usual the convolution with the standard mollifier. As l belongs to $L^\infty(\Omega)$, for every $p < +\infty$, we have $l_\varepsilon \to l$ in $L^p(\Omega)$ as $(\varepsilon \to 0)$. Moreover, $l_\varepsilon \in C^\infty(\Omega)$. Fix now any $\phi \in C_c^1(\Omega)$ with $\phi \geq 0$ on $\{v = \psi\}$ and set $\phi_\varepsilon = \phi \, l_\varepsilon$. We note that ϕ_ε belongs to $C_c^1(\Omega)$ and since $l \geq 0$ also $l_\varepsilon \geq 0$ and thus $\phi_\varepsilon \geq 0$ on $\{v = \psi\}$. Therefore, by (3.37) we have

$$0 \geq \int_\Omega \sqrt{1+|\nabla v^2|}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)\phi_\varepsilon(x) \, \mathrm{d}x \quad \forall \varepsilon > 0.$$

Since $\phi_\varepsilon \to \phi(1+|\nabla v|^2)^{-\frac{1}{2}}$ in $L^2(\Omega)$ as $(\varepsilon \to 0)$, equation (3.38) now follows by passing into this limit and then integrating by parts. Recalling Lemma 2.5.2 we deduce that v is the unique minimal surface above ψ which coincides with u_0 on the boundary. As we will see now, $u(\cdot, t)$ converges to v whenever $t \to +\infty$ and not just along the particular subsequence we just constructed. To verify this, we show that from any sequence of times we can extract another subsequence along which we get convergence to v. Let us therefore start by fixing a sequence of times $s_l \to +\infty$ as $(l \to +\infty)$. Up to possibly passing to a subsequence we can assume that we have $u(\cdot, s_l) \to w$ uniformly, for some $w \in \mathrm{Lip}(\Omega)$. Using Lemma 3.5.2 once more and applying the same arguments as above we claim that we can find a sequence $(t_k)_{k\in\mathbb{N}}$ such that

$$\begin{aligned} &|s_{l_k} - t_k| \leq \frac{1}{k}, \\ &\quad u(\cdot, t_k) \to v \quad \textit{uniformly as } k \to +\infty. \end{aligned}$$

Indeed, for $k \in \mathbb{N}$, we choose $s_{l_k} \geq T_k + 1$. Then Lemma 3.5.2 tells us that

$$\left|\left\{t \in [s_{l_k} - \frac{1}{k}, s_{l_k} + \frac{1}{k}] \ : \ |f(t)| \leq \frac{1}{k}\right\}\right| \geq \frac{1}{k},$$

which allows us to choose $t_k \in [s_{l_k} - \frac{1}{k}, s_{l_k} + \frac{1}{k}]$ such that not only (a), (b) and (c) are satisfied (and thus $u(\cdot, t_k) \to v$), but also $|s_{l_k} - t_k| \leq \frac{1}{k}$. By Corollary 3.3.1 we then have

$$\|u(\cdot, s_{l_k}) - u(\cdot, t_k)\|_{L^\infty(\Omega)} \leq C|s_{l_k} - t_k|^{\frac{1}{2n+2}} \to 0 \quad (k \to +\infty).$$

Since $u(\cdot, s_{l_k}) \to w$ and $u(\cdot, t_k) \to v$, uniformly, we deduce $w = v$. To conclude the proof we need to verify the existence of Φ. Our ansatz for Φ will be

$$\Phi(x,t) := \sum_{k=k_0}^{+\infty} \eta_k(t)(\phi(x) + \xi_k(x)),$$

with

- $\xi_k \in C_c^1(\Omega)$ such that $\xi_k \to 0$ uniformly as $k \to +\infty$,
- $\eta_k \in C_c^1(\mathbb{R})$ with $\eta_k(t'_k) = 1$,

where $(t'_k)_{k \geq k_0}$ is a subsequence of $(t_k)_{k\in\mathbb{N}}$. Moreover, ξ_k and η_k are related in the following way. We assume that there are numbers $\tau_k > 0$, small enough, such that $(]t'_k - \tau_k, t'_k + \tau_k[)_{k\in\mathbb{N}}$ is a family of disjoint intervals and

- $\phi + \xi_k \geq 0$ on $\{x \in \Omega \mid \exists t \in]t'_k - \tau_k, t'_k + \tau_k[:\ u(x,t) = \psi(x,t)\}$,
- $\mathrm{supp}(\eta_k) \subset]t'_k - \tau_k, t'_k + \tau_k[$.

Before we go on with the construction of Φ, let us note that assuming the imposed conditions on ξ_k, η_k and τ_k, it is straightforward to check that Φ belongs to $C^1(\Omega \times]0, +\infty[)$ and satisfies (i), (ii) and (iii). Since for given $\tau_k > 0$, the construction of η_k with the above mentioned properties is standard, we will now focus on the construction of ξ_k. For $k \in \mathbb{N}$ we let $V_k := \{x \in \Omega\ :\ \mathrm{dist}_{\{v=\psi\}}(x) \leq \frac{1}{k\,\mathrm{Lip}(\phi)}\}$ and note that $\phi \geq -\frac{1}{k}$ on V_k. We then fix $k_0 \in \mathbb{N}$, such that $V_{k_0} \subset\subset \Omega$ and set, for $k \geq k_0$: $\alpha(k) := \min_{\overline{\Omega}\setminus V_k}\{v - \psi\}$. Observing that $\alpha(k) > 0$, we deduce that whenever $w : \Omega \to \mathbb{R}$ satisfies $\|w - v\|_{L^\infty(\Omega)} \leq \alpha(k)$ then $\{w = \psi\} \subset V_k$. Consequently, we choose a subsequence $(t'_k)_{k\geq k_0}$ of $(t_k)_{k\in\mathbb{N}}$ such that $\|u(\cdot, t'_k) - v\|_{L^\infty(\Omega)} \leq \frac{1}{2}\alpha(k)$ and, by using the uniform continuity of u in spacetime, $\tau_k > 0$ such that $\|u(\cdot, t) - v\|_{L^\infty(\Omega)} \leq \alpha(k)$, whenever $t \in]t'_k - \tau_k, t'_k + \tau_k[$. Finally, we claim that any $\xi_k \in C_c^1(\Omega)$ with $0 \leq \xi_k \leq \frac{1}{k}$ and $\xi_k \equiv \frac{1}{k}$ in V_{k_0} has the desired properties. Indeed, it is immediate that such ξ_k converges uniformly to zero and by our construction we have

$$\{x \in \Omega \mid \exists t \in]t'_k - \tau_k, t'_k + \tau_k[:\ u(x,t) = \psi(x,t)\} \subset V_k,$$

while $\phi(x) + \xi_k(x) \geq 0$ on V_k. $\square$

4. Viscosity Solutions and Uniqueness

The aim of this chapter is to deduce uniqueness of flat flows, i.e. of the limits of the approximate flows, by using methods from the theory of viscosity solutions. As a byproduct, we will show that our solutions are in particular also viscosity solutions. We will give a precise definition of this notion later. For a more thorough introduction to the theory of viscosity solutions we refer to the so called *user's guide* [35] where parabolic equations are treated in chapter 8. Although parabolic obstacle problems are not mentioned explicitly in this reference, an example therein (namely example 1.7) is dealing with an elliptic obstacle problems and its generalization is straightforward. However, we will collect the relevant definitions and concepts in section 4.1. As before $\Omega \subset \mathbb{R}^n$ is open and bounded, T can either be a positive number or $+\infty$ and $\psi \in C^{1,1}(\Omega)$ represents the obstacle.

4.1. Viscosity Solutions of Parabolic Obstacle Problems

Let us start by recalling the notion of properness which is crucial in the theory of viscosity solutions.

Definition 4.1.1. Let $H : \mathbb{R}^n \times \mathbb{R}^{n\times n}_{sym} \to \mathbb{R}$ be continuous. We say H is *proper* if

$$H(p, Y) \leq H(p, X) \quad \forall p \in \mathbb{R}^n \quad \forall X, Y \in \mathbb{R}^{n\times n}_{sym} : \; X \leq Y,$$

where $\mathbb{R}^{n\times n}_{sym}$, the space of symmetric $n \times n$ matrices is endowed with the usual (partial) ordering, i.e. $X \leq Y$ means

$$\langle (Y - X)v, v\rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

In the following, by a slight abuse of notation, we will use H also as a symbol for the (formal) operator $u \mapsto H(\nabla u, D^2 u)$, where ∇ and D^2 are denoting spacial differentiation.

Definition 4.1.2. Let $V \subset \Omega\times]0, T[$ be open. We say that $u \in C(\overline{\Omega} \times [0, T])$ is a *viscosity subsolution* of the parabolic obstacle problem associated to the operator $\partial_t + H$ and the obstacle ψ in V if for all $(x_0, t_0) \in V$ and every C^2*-test* which is touching u at (x_0, t_0) from above (i.e. for every $\phi \in C^2(\Omega\times]0, T[)$ with $\phi(x_0, t_0) = u(x_0, t_0)$ and $\phi \geq u$ in $\Omega\times]0, T[$) we have

$$\min\left\{\phi_t(x_0, t_0) + H(\nabla\phi(x_0, t_0), D^2\phi(x_0, t_0)), u(x_0, t_0) - \psi(x_0)\right\} \leq 0. \tag{4.1}$$

Similarly, u is called a *viscosity supersolution* of the parabolic obstacle problem associated to the operator $\partial_t + H$ and the obstacle ψ in V if for all $(x_0, t_0) \in V$ and every

C^2*-test* which is touching u at (x_0, t_0) from below (i.e. for every $\phi \in C^2(\Omega\times]0,T[)$ with $\phi(x_0,t_0) = u(x_0,t_0)$ and $\phi \leq u$ in $\Omega\times]0,T[$) we have

$$\min\left\{\phi_t(x_0,t_0) + H(\nabla\phi(x_0,t_0), D^2\phi(x_0,t_0)), u(x_0,t_0) - \psi(x_0)\right\} \geq 0. \tag{4.2}$$

Analogously, in the unconstrained case, we can define the notion of viscosity sub- and supersolutions of the parabolic equation $u_t + H(\nabla u, D^2u) = 0$ if instead of (4.1) we consider the inequalities $\phi_t(x_0,t_0) + H(\nabla\phi(x_0,t_0) \leq 0$ (≥ 0, respectively). Finally, u is called *viscosity solution* if it is both a viscosity sub- and supersolution.

Remark 4.1.1. It can be helpful to note, that in order to check that u is a viscosity subsolution, it suffices to consider only tests ϕ which are touching u at a single point, i.e. with $\phi(x,t) > u(x,t)$ for $(x,t) \neq (x_0,t_0)$. This can be seen as follows: Given $\phi \geq u$ which is touching u possibly also at other points, consider, for $\varepsilon > 0$ the function $\phi_\varepsilon = \phi + \varepsilon(|x-x_0|^2 + |t-t_0|^2)$. First of all, we note that ϕ_ε coincides with u only at (x_0,t_0). Moreover, if we set $F_{(x_0,t_0)}(\phi_\varepsilon) := \partial_t\phi_\varepsilon(x_0,t_0) + H(\nabla\phi_\varepsilon(x_0,t_0), D^2\phi_\varepsilon(x_0,t_0))$, by continuity of H we see that as $\varepsilon \to 0$ we have

$$F_{(x_0,t_0)}(\phi_\varepsilon) \to F_{(x_0,t_0)}(\phi).$$

Therefore, even if a priori we just know that $F_{(x_0,t_0)}(\phi_\varepsilon) \leq 0$ we immediately deduce the same inequality for $F_{(x_0,t_0)}(\phi)$. From this observation the claim follows immediately. Of course, the analog statement holds for checking that u is a supersolution.

Let us now have a closer look at what it means for u to be a viscosity sub- or supersolution of a parabolic obstacle problem. We have the following characterization.

Lemma 4.1.1. *Let $u \in C(\overline{\Omega} \times [0,T])$. Then u is a viscosity supersolution of*

$$\min\left\{u_t + H(\nabla u, D^2u), u - \psi\right\} = 0,$$

(i.e. of the parabolic obstacle problem for $\partial_t + H$ with obstacle ψ) if and only if $u \geq \psi$ and $u_t + H(\nabla u, D^2u) \geq 0$ (in $\Omega\times]0,T[$) in the viscosity sense. Furthermore, u is a viscosity subsolution if and only if $u_t + H(\nabla u, D^2u) \leq 0$ in $\{u > \psi\}$ in the viscosity sense.

Proof. The characterization of supersolutions follows essentially from the definition and the simple fact that the minimum of two numbers is nonnegative if and only if both those numbers are nonnegative. On the other hand, for the characterization of subsolutions we note that in the set $\{u \leq \psi\}$ condition (4.2) is always fulfilled. Consequently, u is a subsolution of the obstacle problem if and only if it is a viscosity subsolution of $u_t + H(\nabla u, D^2u) = 0$ in the complement of this set. □

For the sake of completeness, we check that the operator we consider is indeed proper.

Lemma 4.1.2. *Let $H : \mathbb{R}^n \times \mathbb{R}^{n\times n}_{sym} \to \mathbb{R}$ be defined as*

$$H(p,X) := -\mathrm{tr}(X) + \frac{\langle Xp, p\rangle}{1+|p|^2}. \tag{4.3}$$

Then H is proper.

Proof. Fix $p \in \mathbb{R}^n$, $X, Y \in \mathbb{R}^{n\times n}_{sym}$ with $X \leq Y$. Setting $B := (I_n - \frac{p\otimes p}{1+|p|^2})$, where I_n is the identity matrix in n dimensions, we have

$$H(p, X) = -B : X,$$

where $B : X = \sum_{i,j=1}^n B_{ij}X_{ij}$ is the (Frobenius) inner product of matrices. The properness of H follows once we verified $B : (Y - X) \geq 0$. By assumption, $(Y - X) \geq 0$ and it is easy to see (cf. the proof of Lemma B.1) that all the eigenvalues of B are positive so in particular $B \geq 0$ as well. The claim is now a consequence of the fact that the inner product of two symmetric, non-negative matrices is non-negative. Indeed, let $Z = (X - Y)$ and let $(v_i)_{1\leq i\leq n}$ be a basis of $\mathbb{R}^n$ consisting of eigenvectors of Z. As Z is nonnegative, we thus have $Zv_i = \lambda_i v_i$ for some $\lambda_i \geq 0$. Using the symmetry of Z and the invariance of the trace under coordinate changes we get

$$B : Z = \mathrm{tr}(B \cdot Z) = \sum_{i=1}^n \langle (B \cdot Z)v_i, v_i\rangle = \sum_{i=1}^n \lambda_i \langle Bv_i, v_i\rangle \geq 0.$$

□

4.2. Consistency

In order to show that every flat flow u (i.e. every limit of approximate flows) is also a solution in the viscosity sense, we will employ the following auxiliary result.

Lemma 4.2.1. *Let $\phi \in C^2(\Omega\times]0, T[)$. Then, for every $V \subset \Omega\times]0, T[$ compact we have*

$$\frac{1}{h}\mathrm{sdist}_{\phi(\cdot,t-h)}(x, \phi(x, t)) \to \frac{\phi_t(x, t)}{\sqrt{1 + |\nabla\phi(x, t)|^2}} \quad \textit{in } L^\infty(V) \textit{ as } h \to 0.$$

Proof. Let V be a compact subset of $\Omega\times]0, T[$ and let $h_0 > 0$ small enough such that V_{h_0}, the h_0-neighborhood of V, is still compactly contained in $\Omega\times]0, T[$. Then, for $h < h_0$, let us introduce the auxiliary functions $F_h : V \to \mathbb{R}$ defined as

$$F_h(x, t) := \frac{\frac{1}{h}(\phi(x, t) - \phi(x, t - h))}{\sqrt{1 + |\nabla\phi(x, t - h)|^2}}.$$

As $\phi \in C^2(\Omega\times]0, T[)$ and V is compact it is immediate to check that

$$F_h \to \frac{\phi_t}{\sqrt{1 + |\nabla\phi|^2}} \quad \text{in } L^\infty(V) \text{ as } h \to 0. \tag{4.4}$$

Possibly decreasing the value of $h_0 > 0$, we can assume that for every $h < h_0$

$$d(h) := \left\|\mathrm{dist}_{\phi(\cdot,t-h)}(x, \phi(x, t))\right\|_{L^\infty(V)} < \mathrm{dist}_{\partial\Omega}(x).$$

This allows us to apply Lemma 3.4.5 and deduce that for every $(x,t) \in V$ we have

$$\left|F_h(x,t) - \mathrm{sdist}_{\phi(\cdot,t-h)}(x,\phi(x,t))\right| \leq \\ \mathrm{osc}_{B_{d(h)}(x)}\left(\frac{1}{\sqrt{1+|\nabla\phi(x,t-h)|^2}}\right)\frac{\phi(x,t)-\phi(x,t-h)}{h}. \tag{4.5}$$

The second term on the left hand side is obviously uniformly bounded. Let us argue, why the oscillation term converges to zero uniformly. Since V_{h_0} is compactly contained in $\Omega\times]0,T[$ and since ϕ is of class C^2 for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $(x,t),(y,s) \in V_{h_0}$ we have

$$|x-y| + |t-s| \leq \delta \implies \left|\frac{1}{\sqrt{1+|\nabla\phi(x,t)|^2}} - \frac{1}{\sqrt{1+|\nabla\phi(x,t)|^2}}\right| \leq \varepsilon.$$

In particular, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $(x,t) \in V$ we have

$$\mathrm{osc}_{B_{\delta/2}(x)}\left(\frac{1}{\sqrt{1+|\nabla\phi(x,t-h)|^2}}\right) \leq \varepsilon.$$

Since $d(h) \to 0$ as $h \to 0$ the convergence of the oscillation term follows. This allows us to deduce from (4.5) that

$$\left(F_h(x,t) - \frac{1}{h}\mathrm{sdist}_{\phi(\cdot,t-h)}(x,\phi(x,t))\right) \to 0 \quad \text{in } L^\infty(V) \text{ as } h \to 0. \tag{4.6}$$

Combining (4.4) and (4.6) yields the desired claim.

□

Let $(u^{h_k})_{k\in\mathbb{N}}$ and u be as at the beginning of Section 3.4, i.e. $(u^{h_k})_{k\in\mathbb{N}}$ is a sequence of approximate flows, obtained via Theorem 3.1.3, and for every $T < +\infty$ we have

$$u^{h_k} \to u \quad \text{in } L^\infty(\Omega\times[0,T]) \quad (k \to +\infty).$$

Proposition 4.2.2. *For $(u^{h_k})_{k\in\mathbb{N}}$ and u as above we get that u is a viscosity solution of the parabolic obstacle problem*

$$\min\left\{u_t - \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), u-\psi\right\} = 0, \tag{4.7}$$

in $\Omega\times]0,+\infty[$.

Proof. Recall that u is continuous and satisfies $u \geq \psi$ by construction. Let us first show that u is a viscosity supersolution. According to Lemma 4.1.1 it suffices to show that u solves

$$u_t + H(\nabla u, D^2 u) \geq 0,$$

in the viscosity sense. Fix any point (x_0, t_0) in $\Omega\times]0,+\infty[$ and take an arbitrary ϕ in $C^2(\Omega\times]0,+\infty[)$ with $\phi(x_0,t_0) = u(x_0,t_0)$ and $\phi < u$ in $(\Omega\times]0,+\infty[) \setminus \{(x_0,t_0)\}$ (cf. Remark 4.1.1). By contradiction, suppose that

$$\phi_t(x_0,t_0) - \sqrt{1+|\nabla\phi(x_0,t_0)|^2}\,\mathrm{div}\left(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}\right)(x_0,t_0) < 0.$$

Since ϕ is of class C^2 we can hence find constants $\eta, r > 0$ such that

$$\frac{\phi_t}{\sqrt{1+|\nabla\phi|^2}} \leq \mathrm{div}\left(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}\right) - \eta \quad \forall (x,t) \in Q, \tag{4.8}$$

where $Q := B_r(x_0)\times]t_0 - r, t_0]$. Due to our assumptions on ϕ there exists $\delta > 0$ such that

$$u - \phi > \delta \quad \text{on } \partial_p Q.$$

Let us next choose $k \in \mathbb{N}$ large enough such that for $h := h_k$ we have simultaneously

$$\left\| u^h - u \right\|_{L^\infty(Q)} \leq \delta/4, \tag{4.9}$$

$$\frac{1}{h}\mathrm{sdist}_{\phi(\cdot,t-h)}(x,\phi(x,t)) \leq \frac{\phi_t(x,t)}{\sqrt{1+|\nabla\phi(x,t)|^2}} + \frac{\eta}{4} \quad \forall (x,t) \in Q. \tag{4.10}$$

Note, that this can always be achieved by the uniform convergence of u^{h_k} to u and Lemma 4.2.1. Finally, we set $s := \inf\{\tilde{s} \in \mathbb{R} \,|\, \exists (x,t) \in Q : u^h(x,t) = \phi(x,t) + \tilde{s}\}$. By (4.9) we know that $s \in [-\frac{\delta}{4}, \frac{\delta}{4}]$. Consequently, for $\phi_s = \phi + s$, on $\partial_p Q$ we have

$$u^h - \phi_s = (u^h - u) + (u - \phi) + (\phi - \phi_s) \geq -\frac{\delta}{4} + \delta - \frac{\delta}{4} = \frac{\delta}{2}, \tag{4.11}$$

and in Q, it holds $\phi_s \leq u^h$ (otherwise we get a contradiction to the definition of s).

By definition, we can find a sequence $(s_k)_{k\in\mathbb{N}}$ converging to s with $s_k > s$ for all $k \in \mathbb{N}$ and points $(x_k,t_k) \in Q$ with $u^h(x_k,t_k) = \phi_{s_k}(x_k,t_k)$. Possibly passing to a subsequence, we can assume that (x_k,t_k) converges to some $(\bar{x},\bar{t}) \in \overline{Q}$. Either we have $u^h(\bar{x},\bar{t}) = \phi_s(\bar{x},\bar{t})$ or there is a subsequence $(t'_k)_{k'\in\mathbb{N}}$ of $(t_k)_{k\in\mathbb{N}}$ with the property that $t'_k < \bar{t}$ (otherwise Remark 3.2.1 yields a contradiction). Note that this second case can only appear at times $\bar{t} = Nh$ for some $N \in \mathbb{N}$. In this latter case we will consider the function $u^{h,-} : \overline{\Omega}\times]0,+\infty[\to \mathbb{R}$ which we define as follows: If $k \in \mathbb{N}_0$ is such that $t \in]kh,(k+1)h]$ then we set

$$u^{h,-}(x,t) := u_k(x).$$

Note that u^h and $u^{h,-}$ will only differ at times Nh. Moreover, now it holds $u^{h,-}(\bar{x},\bar{t}) = \phi_s(\bar{x},\bar{t})$. By (4.11) – which also holds if we replace u^h by $u^{h,-}$– we see that $(\bar{x},\bar{t}) \notin \partial_p Q$ and hence there exists $\rho > 0$ such that $B_\rho(\bar{x}) \subset B_r(x_0)$. In the remainder of the proof, for convenience, we will assume that we are in the case $u^h(\bar{x},\bar{t}) = \phi(\bar{x},\bar{t})$. However, it is straightforward to check that the inequalities (4.12) and (4.13) below can be derived analogously for $u^{h,-}$.

Since $u^h(\cdot,\bar{t})$ solves a variational inequality, by Proposition 2.3.4 we know that for a.e. $x \in B_\rho(\bar{x})$

$$\operatorname{div}\left(\frac{\nabla u^h(\cdot,\bar{t})}{\sqrt{1+|\nabla u^h(\cdot,\bar{t})|^2}}\right)(x) \le \frac{1}{h}\operatorname{sdist}_{u^h(\cdot,\bar{t}-h)}(x,u^h(x,\bar{t})). \tag{4.12}$$

Since $u^h(\bar{x},\bar{t}) = \phi_s(\bar{x},\bar{t})$ and as the maps

$$\begin{aligned} B_\rho(\bar{x}) \ni x &\mapsto \operatorname{sdist}_{u^h(\cdot,\bar{t}-h)}(x,\phi_s(x,\bar{t})) \in \mathbb{R}, \\ B_\rho(\bar{x}) \ni x &\mapsto \operatorname{sdist}_{u^h(\cdot,\bar{t}-h)}(x,u^h(x,\bar{t})) \in \mathbb{R}, \end{aligned}$$

are continuous, there exists $0 < \rho_1 \le \rho$ such that

$$\operatorname{sdist}_{u^h(\cdot,\bar{t}-h))}(x,u^h(x,\bar{t})) \le \operatorname{sdist}_{u^h(\cdot,\bar{t}-h)}(x,\phi_s(x,\bar{t})) + \frac{\eta}{4} \quad \forall x \in B_{\rho_1}(\bar{x}).$$

By construction, $\phi_s \le u^h$. Hence Lemma 1.2.1 and the above inequality imply

$$\begin{aligned} \operatorname{sdist}_{u^h(\cdot,\bar{t}-h)}(x,u^h(x,\bar{t})) &\le \operatorname{sdist}_{\phi_s(\cdot,\bar{t}-h)}(x,\phi_s(x,\bar{t})) + \frac{\eta}{4} \\ &= \operatorname{sdist}_{\phi(\cdot,\bar{t}-h)}(x,\phi(x,\bar{t})) + \frac{\eta}{4} \quad \forall x \in B_{\rho_1}(\bar{x}). \end{aligned} \tag{4.13}$$

Combining (4.8), (4.10), (4.12) and (4.13) we finally get

$$\operatorname{div}\left(\frac{\nabla u^h(\cdot,\bar{t})}{\sqrt{1+|\nabla u^h(\cdot,\bar{t})|^2}}\right) \le \operatorname{div}\left(\frac{\nabla \phi(\cdot,\bar{t})}{\sqrt{1+|\nabla \phi(\cdot,\bar{t})|^2}}\right) - \frac{\eta}{2} \quad \text{a.e. in } B_{\rho_1}(\bar{x}).$$

It is now easy to deduce a contradiction by using the comparison principle. Indeed, let $\varepsilon > 0$ be small enough such that for $v(x) := u^h(x,\bar{t})$ and $w(x) := \phi(x,\bar{t}) - \varepsilon|x-\bar{x}|^2$ we have

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) \le \operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) \quad \text{a.e. in } B_{\rho_1}(\bar{x}).$$

Since $v(\bar{x}) = w(\bar{x})$ and as on $\partial B_{\rho_1}(\bar{x})$ we know that $v-w \ge \varepsilon\rho_1^2$, the desired contradiction now follows by the maximum principle (cf. Proposition A.6), namely

$$0 = \inf_{B_{\rho_1}(\bar{x})}(v-w) = \inf_{\partial B_{\rho_1}(\bar{x})}(v-w) > 0.$$

What remains to be shown is that u is also a viscosity subsolution in the non-coincidence set $\{u > \psi\}$. This can be done in a similar fashion. The only place, in which we need to adapt the proof, is the counterpart of inequality (4.12). More precisely, let us consider some $(x_0, t_0) \in \{u > \psi\}$ and ϕ in $C^2(\Omega\times]0, +\infty[)$ with $\phi(x_0, t_0) = u(x_0, t_0)$ and $\phi > u$ in $(\Omega\times]0, +\infty[) \setminus \{(x_0, t_0)\}$. By contradiction, suppose now that

$$\phi_t(x_0, t_0) - \sqrt{1 + |\nabla\phi(x_0, t_0)|^2}\operatorname{div}\left(\frac{\nabla\phi}{\sqrt{1 + |\nabla\phi|^2}}\right)(x_0, t_0) > 0.$$

As before, we now choose $\eta, r > 0$ small enough such that we have

$$\frac{\phi_t}{\sqrt{1 + |\nabla\phi|^2}} \geq \operatorname{div}\left(\frac{\nabla\phi}{\sqrt{1 + |\nabla\phi|^2}}\right) + \eta \quad \forall(x, t) \in Q,$$

where again $Q = B_r(x_0)\times]t_0 - r, t_0]$ but this time we additionally require r to be small enough such that $Q \subset\subset \{u > \psi\}$. This can always be done as $\{u > \psi\}$ is open. Consequently, we can find $\delta > 0$ which is not only satisfying $\phi - u > \delta$ on $\partial_p Q$ but also $u - \psi > \delta$ in Q. Having made these refinements the proof now follows exactly the same ideas as above. Note in particular that the sole purpose of the new requirements is to guarantee that u^h stays above the obstacle in Q which then allows to deduce that for given $\bar{t} \in]t_0 - r, t_0]$

$$\operatorname{div}\left(\frac{\nabla u^h(\cdot, \bar{t})}{\sqrt{1 + |\nabla u^h(\cdot, \bar{t})|^2}}\right)(x) = \frac{1}{h}\operatorname{sdist}_{u^h(\cdot, \bar{t}-h)}(x, u^h(x, \bar{t})),$$

for almost every $x \in B_r(x_0)$. □

4.3. The Comparison Principle for Viscosity Solutions

The notion of viscosity sub- and supersolutions allows us now to prove the following comparison principle which can be seen as a variant of Theorem 8.2 in [35].

Proposition 4.3.1. *Let u be a viscosity subsolution and v be a viscosity supersolution of* (4.7) *such that $u \leq v$ on $\partial_p(\Omega \times [0, T[)$. Then $u \leq v$ in $\Omega \times [0, T[$.*

Before we come to the proof, let us state a result which will be used in an essential way to derive the comparison principle.

Lemma 4.3.2 (Doubling of Variables)**.** *Let $u, v \in C(\overline{\Omega} \times [0, T[)$ such that $u(x, t) \to -\infty$ uniformly in $\overline{\Omega}$ as $t \to T$ and such that v is bounded from below. Then for $\alpha > 0$ suppose that at $(\hat{x}, \hat{y}, \hat{t}) \in \Omega \times \Omega\times]0, T[$ the function*

$$\overline{\Omega} \times \overline{\Omega} \times [0, T[\ni (x, y, t) \mapsto \Phi_\alpha(x, y, t) := u(x, t) - v(y, t) - \frac{\alpha}{2}|x - y|^2,$$

attains its maximum $M_\alpha := \Phi_\alpha(\hat{x}, \hat{y}, \hat{t})$ *and* $M_\alpha > 0$.

Moreover, we assume that there exists $r > 0$ *such that for every* $M > 0$ *there exists* $K > 0$ *with the following property: For every* $(x, y, t) \in B_r(\hat{x}, \hat{y}, \hat{t})$ *we get that*

a) if $\phi \in C^2(\Omega\times]0,T[)$ *with* $\phi \geq u$, $\phi(x,t) = u(x,t)$ *and* $|\nabla\phi(x,t)| + \left|D^2\phi(x,t)\right| \leq M$, *then we have* $\phi_t(x,t) \leq K$,

b) if $\phi \in C^2(\Omega\times]0,T[)$ *with* $\phi \leq v$, $\phi(y,t) = v(y,t)$ *and* $|\nabla\phi(y,t)| + \left|D^2\phi(y,t)\right| \leq M$, *then we have* $\phi_t(y,t) \geq -K$.

Then we can find two sequences $(x_k, t_k)_{k\in\mathbb{N}}, (y_k, s_k)_{k\in\mathbb{N}}$ *in* $\Omega\times]0,T[$ *converging to* $(\hat{x}, \hat{t})$ *and* $(\hat{y}, \hat{t})$ *and two sequences* $(\phi_{u,k})_{k\in\mathbb{N}}$ *and* $(\phi_{v,k})_{k\in\mathbb{N}}$ *in* $C^2(\Omega\times]0,T[)$, $a \in \mathbb{R}$, $p \in \mathbb{R}^n$ *and* $X \leq Y \in \mathbb{R}^{n\times n}_{sym}$ *with the following properties*

i) $\phi_{u,k} \geq u$ *and* $\phi_{v,k} \leq v$ *in* $\Omega \times [0,T[\quad \forall k \in \mathbb{N}$,

ii) $\phi_{u,k}(x_k,t_k) = u(x_k,t_k)$ *and* $\phi_{v,k}(y_k,s_k) = v(y_k,s_k) \quad \forall k \in \mathbb{N}$,

iii) $\partial_t\phi_{u,k}(x_k,t_k) \to a$ *and* $\partial_t\phi_{v,k}(y_k,s_k) \to a$ *as* $(k \to \infty)$,

iv) $\nabla\phi_{u,k}(x_k,t_k) \to \alpha(\hat{x}-\hat{y})$ *and* $\nabla\phi_{v,k}(y_k,s_k) \to \alpha(\hat{x}-\hat{y})$ *as* $(k \to \infty)$,

v) $D^2\phi_{u,k}(x_k,t_k) \to X$ *and* $D^2\phi_{v,k}(y_k,s_k) \to Y$ *as* $(k \to \infty)$.

We postpone the proof of this technical but crucial until after the proof of the comparison principle which we will give now.

Proof (of Proposition 4.3.1, assuming Lemma 4.3.2). Of course, the case $T = +\infty$ follows once we derived the proposition for every finite T. Let us therefore assume that $T < +\infty$. For reasons to be explained later, let us replace the function u by

$$\widetilde{u}(x,t) := u(x,t) - \frac{\varepsilon}{T-t}, \quad (x,t) \in \overline{\Omega} \times [0,T[,$$

for $\varepsilon > 0$ arbitrary. It is then easy to check that there exists a constant $c = c(\varepsilon, T) > 0$ such that $\widetilde{u}$ is a viscosity subsolution of

$$\min\left\{u_u - \sqrt{1+|\nabla u|^2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), u - \psi\right\} + c = 0, \tag{4.14}$$

in $\Omega\times]0,T[$. Additionally, we have

$$u(x,t) \to -\infty \tag{4.15}$$

uniformly in $\overline{\Omega}$ as $t \to T$. We will deliver a verification of these assertions at the end of the proof. Obviously, if we manage to prove that $\widetilde{u} \leq v$ in $\overline{\Omega} \times [0,T[$, then the desired inequality for u follows by arbitrariness of $\varepsilon > 0$.

By a slight abuse of notation we will henceforth write again u instead of $\widetilde{u}$. We assume by contradiction that there exists some $(z,t) \in \Omega \times]0,T[$ and $\delta > 0$ such that

$$u(z,t) - v(z,t) = \delta. \tag{4.16}$$

For $\alpha > 0$ to be fixed momentarily, let us consider $\Phi_\alpha : \overline{\Omega} \times \overline{\Omega} \times [0,T[\to \mathbb{R}$, defined via

$$\Phi_\alpha(x,y,t) := u(x,t) - v(y,t) - \frac{\alpha}{2}|x-y|^2.$$

We note that as Φ_α is bounded from above, its supremum is finite. Moreover, Φ_α is continuous on $\overline{\Omega} \times [0,\tau]$, for any $\tau < T$. Hence by compactness of $\overline{\Omega}$ and in view of (4.15) Φ_α attains its supremum at a point $(x_\alpha, y_\alpha, t_\alpha) \in \overline{\Omega} \times \overline{\Omega} \times [0,T[$. Since $\Phi(z,z,s) = \delta$ we get

$$0 < \delta \le M_\alpha := \Phi_\alpha(x_\alpha, y_\alpha, t_\alpha),$$

Note that this implies in particular that

$$0 < u(x_\alpha, t_\alpha) - v(y_\alpha, t_\alpha). \tag{4.17}$$

Towards applying the previous lemma, let us next rule out the case that either x_α or y_α belong to $\partial\Omega$ or that $t_\alpha = 0$. Note, first of all, that for $\alpha' > \alpha$ we have $M_{\alpha'} \le M_\alpha$. Therefore, since $M_\alpha > \delta$ we know that $\lim_{\alpha \to +\infty} M_\alpha$ exists and is positive. From the definitions of $M_{\frac{\alpha}{2}}$ and $(x_\alpha, y_\alpha, t_\alpha)$ we get

$$M_{\frac{\alpha}{2}} \ge u(x_\alpha, t_\alpha) - v(y_\alpha, t_\alpha) - \frac{\alpha}{4}|x_\alpha - y_\alpha|^2 = M_\alpha + \frac{\alpha}{4}|x_\alpha - y_\alpha|^2,$$

and consequently, since $(M_\alpha)_{\alpha>0}$ converges as $(\alpha \to +\infty)$, we can deduce

$$\frac{\alpha}{4}|x_\alpha - y_\alpha|^2 \to 0, \tag{4.18}$$

as $(\alpha \to +\infty)$. Suppose that $x_\alpha \in \partial\Omega$. By assumption we have $u(x_\alpha, t_\alpha) \le v(x_\alpha, t_\alpha)$. Hence by continuity of v and by (4.18), for α large enough, we get $u(x_\alpha, t_\alpha) \le v(y_\alpha, t_\alpha)$ which would imply that $M_\alpha \le 0$ which is a contradiction. Analogously, we can rule out that $y_\alpha \in \partial\Omega$ or that $t_\alpha = 0$ if α is chosen large enough. Thus, for such an α we denote by $(\hat{x}, \hat{y}, \hat{t}) \in \Omega \times \Omega \times]0,T[$ a maximal point of Φ_α.

Before we can apply the lemma we also need to check the condition on the time derivative of certain test functions. Let us start by choosing $r > 0$ small enough such that $u(x,t) > \psi(x,t)$ whenever $(x,y,t) \in B_r(\hat{x}, \hat{y}, \hat{t})$. This can be done since we know that $\delta < u(\hat{x}, \hat{t}) - v(\hat{y}, \hat{t})$ and that $|\hat{x} - \hat{y}| \to 0$ as $(\alpha \to \infty)$. Thus, by possibly enlarging α further, we can guarantee that first of all $u(\hat{x}, \hat{t}) > v(\hat{x}, \hat{t})$ and then that there exists some $r > 0$ with the property that $u(x,t) > v(x,t)$ whenever $(x,y,t) \in B_r(\hat{x}, \hat{y}, \hat{t})$. In particular, since v is a supersolution and thus has to stay above the obstacle, we get that $u(x,t) > \psi(x,t)$ whenever $(x,t) \in B_r(\hat{x}, \hat{t})$. Consider now first a function $\phi \in C^2(\Omega \times]0,T[)$ touching u from above at some point $(x,t) \in B_r(\hat{x}, \hat{t})$. By construction (and assumption) we know that

$$\min\left\{ \phi_t - \sqrt{1+|\nabla\phi|^2}\,\mathrm{div}\left(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}\right), \phi - \psi \right\}(x,t) \le -c < 0.$$

By the choice of r, we can deduce that $\phi_t(x,t) + H(\nabla\phi(x,t), D^2\phi(x,t)) \leq 0$. Assuming that $|\nabla\phi(x,t)| + \left|D^2\phi(x,t)\right| \leq M$ we can thus show the existence of a constant $K = K(n,M)$ such that

$$\phi_t(x,t) \leq \left|H(\nabla\phi(x,t), D^2\phi(x,t))\right| \leq K,$$

where here and in the following H is defined according to (4.3). On the other hand, consider now $\phi \in C^2(\Omega\times]0,T[)$ touching v from below at some point $(y,t) \in B_r(\hat{y},\hat{t})$. Since v is a supersolution we know right away that $\phi_t(y,t) + H(\nabla\phi(y,t), D^2\phi(y,t)) \geq 0$ and we can conclude as before. Thus, we are now in the position to apply Lemma 4.3.2 to get $(\phi_{u,k})_{k\in\mathbb{N}}$, $(\phi_{v,k})_{k\in\mathbb{N}}$ in $C^2(\Omega\times]0,T[)$, $(x_k,t_k)_{k\in\mathbb{N}}, (y_k,s_k)_{k\in\mathbb{N}}$ in $\Omega\times]0,T[$, $a \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $X \leq Y \in \mathbb{R}^{n\times n}_{sym}$, as in the lemma. Note that by properties $i)$ and $ii)$ in Lemma 4.3.2 we get $P_{\phi_{u,k}}(x_k,t_k) \leq -c$ and $P_{\phi_{v,k}}(y_k,s_k) \geq 0$ for all $k \in \mathbb{N}$, where

$$P_\phi(x,t) := \min\left\{\phi_t - \sqrt{1+|\nabla\phi|^2}\operatorname{div}\left(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}\right), \phi - \psi\right\}(x,t).$$

Using $iii), iv)$ and $v)$ in Lemma 4.3.2 we can pass into the limit ($k \to \infty$) in these two relations to deduce

$$P_1 := \min\left\{a + H(\alpha(\hat{x}-\hat{y}), X), u(\hat{x},\hat{t}) - \psi(\hat{x})\right\} \leq -c, \tag{4.19}$$

$$P_2 := \min\left\{a + H(\alpha(\hat{x}-\hat{y}), Y), v(\hat{y},\hat{t}) - \psi(\hat{y})\right\} \geq 0. \tag{4.20}$$

Subtracting the two inequalities yields

$$c \leq P_2 - P_1.$$

We claim that this already constitutes a contradiction for α large enough. Indeed, let us consider the various possible cases:

Case 1: $P_1 = a + H(\alpha(\hat{x}-\hat{y}), X)$ and $P_2 = a + H(\alpha(\hat{x}-\hat{y}), Y)$.
In this case, we get by Lemma 4.1.2 that

$$c \leq H(\alpha(\hat{x}-\hat{y}), Y) - H(\alpha(\hat{x}-\hat{y}), X) \leq 0,$$

which is clearly a contradiction as $c > 0$.

Case 2: $P_1 = u(\hat{x},\hat{t}) - \psi(\hat{x})$ and $P_2 = v(\hat{y},\hat{t}) - \psi(\hat{y})$.
Using (4.17) we deduce

$$c \leq v(\hat{y},\hat{t}) - \psi(\hat{y}) - u(\hat{x},\hat{t}) + \psi(\hat{x}) < \psi(\hat{x}) - \psi(\hat{y}).$$

Letting α be large enough, thanks to (4.18) and the uniform continuity of ψ we can make the right hand side arbitrarily small and thus deduce a contradiction.

Case 3: $P_1 = u(\hat{x},\hat{t}) - \psi(\hat{x})$ and $P_2 = a + H(\alpha(\hat{x}-\hat{y}), Y)$.
This case can easily be reduced to the second one since $P_2 \leq v(\hat{y},\hat{t}) - \psi(\hat{y})$ so that

$$c \leq P_2 - P_1 \leq v(\hat{y},\hat{t}) - \psi(\hat{y}) - u(\hat{x},\hat{t}) + \psi(\hat{x}),$$

and we can argue as before.

Case 4: $P_1 = a + H(\alpha(\hat{x}-\hat{y}), X)$ and $P_2 = v(\hat{y},\hat{t}) - \psi(\hat{y})$.
Using $P_2 \le a + H(\alpha(\hat{x}-\hat{y}), Y)$ we can argue as in the first case. Thus we showed that inequality (4.16) leads to a contradiction.

Finally let us discuss the assertions made at the very beginning of this proof. For $\varepsilon > 0$ we define $\widetilde{u} := u - \frac{\varepsilon}{T-t}$. First of all it is clear that as $t \to T$, we get $u(x,t) \to -\infty$ uniformly. Moreover, observe that u is bounded since by definition u is continuous and the set $\overline{\Omega} \times [0,T]$ compact. Hence u_ε is bounded from above. Then, let us fix $(x_0,t_0) \in \Omega \times]0,T[$ and $\widetilde{\phi} \in C^2(\Omega \times]0,T[)$ touching $\widetilde{u}$ at (x_0,t_0) from above. Note that this implies that $\phi := \widetilde{\phi} + \frac{\varepsilon}{T-t}$ is touching u at (x_0,t_0) from above. Now we have to distinguish between two cases. Let us first consider the case $u(x_0,t_0) \le \psi(x_0)$. We get

$$\widetilde{\phi}(x_0,t_0) - \psi(x_0) \le -\frac{\varepsilon}{T-t_0} \le -\frac{\varepsilon}{T}.$$

On the other hand, if $u(x_0,t_0) > \psi(x_0)$, we know (as u is a viscosity of (4.7)) that

$$\phi_t(x_0,t_0) - \sqrt{1+|\nabla\phi(x_0,t_0)|^2}\,\mathrm{div}\left(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}\right)(x_0,t_0) \le 0,$$

from which we immediately deduce

$$\widetilde{\phi}_t(x_0,t_0) - \sqrt{1+\left|\nabla\widetilde{\phi}(x_0,t_0)\right|^2}\,\mathrm{div}\left(\frac{\nabla\widetilde{\phi}}{\sqrt{1+|\nabla\widetilde{\phi}|^2}}\right)(x_0,t_0) \le -\frac{\varepsilon}{(T-t_0)^2} \le -\frac{\varepsilon}{T^2}.$$

Setting $c = \min\left\{\frac{\varepsilon}{T}, \frac{\varepsilon}{T^2}\right\}$, this shows that $\widetilde{u}$ is a viscosity subsolution of (4.14). Thus we verified that $\widetilde{u}$ satisfies the two additional assumptions made in the beginning of the proof. $\square$

We will derive now Lemma 4.3.2 from a similar result (in an elliptic setting) which is given in [35]. The following notion turns out to be useful in what follows.

Definition 4.3.1. For $V \subset \mathbb{R}^N$ open, $u \in C(V)$ and $x_0 \in V$ we define the set of *second order superjets (of u at x_0)* as

$$J^{2,+}u(x_0) := \{(p,X) \in \mathbb{R}^N \times \mathbb{R}^{N\times N}_{sym} : \text{ as } x \to x_0 \text{ we have}$$
$$u(x) \le u(x_0) + \langle p, x-x_0\rangle + \frac{1}{2}\langle X(x-x_0), x-x_0\rangle + o(|x-x_0|^2)\ \}.$$

Analogously, we define the *second order subjets (of u at x_0)* as

$$J^{2,-}u(x_0) := \{(p,X) \in \mathbb{R}^N \times \mathbb{R}^{N\times N}_{sym} : \text{ as } x \to x_0 \text{ we have}$$
$$u(x) \ge u(x_0) + \langle p, x-x_0\rangle + \frac{1}{2}\langle X(x-x_0), x-x_0\rangle + o(|x-x_0|^2)\ \},$$

and we notice that $J^{2,-}u(x_0) = -J^{2,+}(-u)(x_0)$.

For a more detailed discussion of this notion, we refer the unacquainted reader to the last part of appendix B. In the application below we will treat the time variable as an additional space variable and thus we work in $n+1$ dimensions, hence the choice of the letter N which denotes the dimension in the definition above. Moreover, the link between the elliptic and parabolic setting is formulated in the following simple lemma. In order to make the connection more obvious, we will label the x_{n+1} coordinate by t in the following lemma.

Lemma 4.3.3. *Let $V \subset \mathbb{R}^{n+1}$ open, $u \in C(V)$, $(\hat{x}, \hat{t}) \in V \subset \mathbb{R}^n \times \mathbb{R}$ and finally $(p, X) \in J^{2,+}u(\hat{x}, \hat{t})$. Then there exists $\phi \in C^2(V)$ such that*

i) $\phi \geq u$ *in* V,

ii) $\phi(\hat{x}, \hat{t}) = u(\hat{x}, \hat{t})$,

iii) $\phi_t(\hat{x}, \hat{t}) = p_{n+1}$,

iv) $\nabla_x \phi(\hat{x}, \hat{t}) = p'$,

v) $D_x^2 \phi(\hat{x}, \hat{t}) = X'$,

where $p = (p', p_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ and X' denotes the $n \times n$-submatrix obtained from X by deleting the last row and the last column.

Proof. Let $(x, t) \in V$. By definition we know that as $(x, t) \to (\hat{x}, \hat{t})$ we have for $X = (X_{ij})$

$$\begin{aligned} u(x,t) \leq u(\hat{x}, \hat{t}) &+ \langle p', x - \hat{x} \rangle + p(t - \hat{t}) + \frac{1}{2} \langle X'(x - \hat{x}), x - \hat{x} \rangle \\ &+ \sum_{k=1}^{n} X_{1k}(x_k - \hat{x}_k)(t - \hat{t}) + \frac{1}{2} X_{(n+1)(n+1)}(t - \hat{t})^2 + o(|x - \hat{x}|^2 + \left|t - \hat{t}\right|^2). \end{aligned}$$

As in the proof of Lemma B.10 we can replace $o(|x - \hat{x}|^2 + \left|t - \hat{t}\right|^2)$ by a C^2-function $h(x, t)$ with vanishing at $(\hat{x}, \hat{t})$ up to the second order derivatives and set

$$\begin{aligned} \phi(x,t) := u(\hat{x}, \hat{t}) &+ \langle p', x - \hat{x} \rangle + p(t - \hat{t}) + \frac{1}{2} \langle X'(x - \hat{x}), x - \hat{x} \rangle \\ &+ \sum_{k=1}^{n} X_{1k}(x_k - \hat{x}_k)(t - \hat{t}) + \frac{1}{2} X_{(n+1)(n+1)}(t - \hat{t})^2 + o(|x - \hat{x}|^2 + \left|t - \hat{t}\right|^2). \end{aligned}$$

It is then straightforward to check that ϕ satisfies properties *i)* to *v)*.

□

With the notion of sub- and superjets at hand, we now recall the following result which corresponds to [35, Theorem 3.2].

Proposition 4.3.4. *Let $V \subset \mathbb{R}^N$ open and bounded, $u, v \in C(V)$ and $\phi \in C^2(\mathbb{R}^N \times \mathbb{R}^N)$. Suppose $(\hat{x}, \hat{y}) \in V \times V$ is a maximum of*

$$V \times V \ni (x, y) \mapsto u(x) - v(y) - \phi(x, y).$$

Then there exists $X, Y \in \mathbb{R}^{N \times N}_{sym}$ such that

$$(\partial_x \phi(\hat{x}), X) \in \overline{J^{2,+}u(\hat{x})} \quad \text{and} \quad (-\partial_y \phi(\hat{y}), Y) \in \overline{J^{2,-}v(\hat{y})},$$

and the block diagonal matrix with entries X, $-Y$ satisfies

$$-(1 + \|A\|)I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + A^2, \tag{4.21}$$

where $A = D^2\phi(\hat{x}, \hat{y})$ and I is the identify matrix of dimension $2N$.

A detailed discussion of the proof of this result can be found in the Appendix of [35]. Taking this proposition for granted, we can now proof Lemma 4.3.2.

Proof of Lemma 4.3.2. Let us first of all consider the case that Φ_α attains a strict, global maximum at $(\hat{x}, \hat{y}, \hat{t})$. Then, for $\beta > 0$ we consider the function

$$\begin{aligned} \Psi_\beta : \overline{\Omega} \times \overline{\Omega} \times [0, T[\times[0, T[\quad &\to \quad \mathbb{R}, \\ (x, y, t, s) \quad &\mapsto \quad u(x, t) - v(y, s) - \frac{\alpha}{2}|x - y|^2 - \frac{\beta}{2}(t - s)^2. \end{aligned}$$

By the assumptions on u and v we see that Ψ_β attains its supremum at some point $P_\beta = (x_\beta, y_\beta, t_\beta, s_\beta) \in \overline{\Omega} \times \overline{\Omega} \times [0, T[\times[0, T[$. Let $N_\beta = \sup \Psi_\beta$. Using $(\hat{x}, \hat{y}, \hat{t}, \hat{t})$ as a comparison point we see that $N_\beta \geq M_\alpha$ for every $\beta > 0$. Analogous to the derivation of (4.18) we can show that $|t_\beta - s_\beta| \to 0$ as $(\beta \to +\infty)$. By the assumption that Φ_α has a unique, strict maximum we can also see that

$$P_\beta \to (\hat{x}, \hat{y}, \hat{t}, \hat{t}). \tag{4.22}$$

Indeed, assume by contradiction that there exists a $\rho > 0$ such that for all β_0 there exists a $\beta \geq \beta_0$ with $\left|(x_\beta, y_\beta, t_\beta, s_\beta) - (\hat{x}, \hat{y}, \hat{t}, \hat{t})\right| > \rho$. Then, by the strictness of the maximum of Φ_α, there exists some $\delta > 0$ such that

$$u(x_\beta, t_\beta) - v(y_\beta, t_\beta) - \frac{\alpha}{2}|x_\beta - y_\beta|^2 \leq M_\alpha - 2\delta.$$

Hence by continuity of v and by possibly increasing β_0 we can find $\beta > \beta_0$ such that

$$N_\beta \leq u(x_\beta, t_\beta) - v(y_\beta, s_\beta) - \frac{\alpha}{2}|x_\beta - y_\beta|^2 \leq M_\alpha - \delta,$$

which is a contradiction. Now we apply Proposition 4.3.4 with $N = n+1$ where we treat the time variable as another space variable, this means that we think of $t = x_{n+1}$ and

$s = y_{n+1}$. Nevertheless, we continue to write s and t in the following, so that we apply the proposition to

$$\phi(x,t,y,s) := \frac{\alpha}{2}|x-y|^2 - \frac{\beta}{2}(t-s)^2.$$

Noting that $\nabla_x \phi(x,t,y,s)) = \alpha(x-y)$, $-\nabla_y \phi_\alpha(x,t,y,s) = \alpha(x-y)$ and $\partial_t \phi(x,t,y,s) = -\partial_s \phi(x,t,y,s) = \beta(t_\beta - s_\beta)$, the proposition yields elements

$$(\alpha(x_\beta - y_\beta), \beta(t_\beta - s_\beta), \widetilde{X}_\beta) \in \overline{J^{2,+}u(x_\beta, t_\beta)}, \tag{4.23}$$

$$(\alpha(x_\beta - y_\beta), \beta(t_\beta - s_\beta), \widetilde{Y}_\beta) \in \overline{J^{2,-}v(y_\beta, s_\beta)}, \tag{4.24}$$

where $\widetilde{X}_\beta$ and $\widetilde{Y}_\beta$ satisfy a specific structural condition which we will exploit now. Note that for $A = D^2\phi(\hat{x}, \hat{t}, \hat{y}, \hat{s}) \in \mathbb{R}^{2(n+1)\times 2(n+1)}$ we get

$$A + A^2 = \begin{pmatrix} (\alpha+2\alpha^2)I_n & 0 & -(\alpha+2\alpha^2)I_n & 0 \\ 0 & \beta+2\beta^2 & 0 & -\beta-2\beta^2 \\ (\alpha+2\alpha^2)I_n & 0 & -(\alpha+2\alpha^2)I_n & 0 \\ 0 & \beta+2\beta^2 & 0 & -\beta-2\beta^2 \end{pmatrix},$$

where I_n is the identify matrix in $\mathbb{R}^n$. Therefore, $A + A^2$ vanishes on vectors of the form $(v, 0, v, 0)$, where $v \in \mathbb{R}^n$. Relation (4.21) can therefore be used to deduce that

$$\langle (v,0), \widetilde{X}_\beta(v,0)\rangle \leq \langle (v,0), \widetilde{Y}_\beta(v,0)\rangle,$$

or, in other words, that $X_\beta \leq Y_\beta$ where X_β and Y_β are the $n \times n$ submatrices obtained from deleting the $(n+1)$-th row and the $(n+1)$-th column of the matrix $\widetilde{X}_\beta$ and $\widetilde{Y}_\beta$ respectively. We note furthermore, that due to (4.21) we also know that the matrices X_β and Y_β lie in a bounded set. We can therefore pass to a subsequence $\beta_k \to \infty$ as $k \to \infty$ such that $X_{\beta_k} \to X$ and $Y_{\beta_k} \to Y$ as $k \to \infty$ for some symmetric matrices X, Y. Note that the relation $X_\beta \leq Y_\beta$ allows to pass into the limit so that we have $X \leq Y$. Moreover, we recall that $(x_{\beta_k}, t_{\beta_k}, y_{\beta_k}, s_{\beta_k}) \to (\hat{x}, \hat{t}, \hat{y}, \hat{s})$ as $k \to \infty$. We will henceforth write x_k instead of x_{β_k} etc. Due to Lemma 4.3.3 we now get for each β_k two sequences of functions $(\phi^k_{u,l})_{l\in\mathbb{N}}$ and $(\phi^k_{v,l})_{l\in\mathbb{N}}$ in $C^2(\Omega\times]0,T[)$ with the following properties:

i) $\phi^k_{u,l} \geq u$ and $\phi^k_{v,l} \leq v$ in $\Omega\times]0,T[$ for every $l \in \mathbb{N}$,

ii) $\phi^k_{u,l}(x_k, t_k) = u(x_k, t_k)$ and $\phi^k_{v,l}(y_k, s_k) = u(y_k, s_k)$,

iii) $\partial_t \phi^k_{u,l}(x_k, t_k) \to \beta_k(t_k - s_k)$ and $\partial_t \phi^k_{v,l}(y_k, s_k) \to \beta_k(t_k - s_k)$ as $l \to \infty$,

iv) $\nabla_x \phi^k_{u,l}(x_k, t_k) \to \alpha(x_k - y_k)$ and $\nabla_x \phi^k_{v,l}(y_k, s_k) \to \alpha(x_k - y_k)$ as $l \to \infty$,

v) $D^2_x \phi^k_{u,l}(x_k, t_k) \to X_{\beta_k}$ and $D^2_x \phi^k_{v,l}(y_k, s_k) \to Y_{\beta_k}$ as $l \to \infty$.

According to (4.22) let now $k_0 \in \mathbb{N}$ be large enough so that $(x_k, t_k, y_k, s_k) \in B_r(\hat{x}, \hat{t}, \hat{y}, \hat{t})$ for all $k \geq k_0$ and where r is as in hypothesis of the lemma we are proving. Since each of the four sequences $(\nabla\phi^k_{u,l}(x_k, t_k))_{l\in\mathbb{N}}$, $(\nabla\phi^k_{u,l}(x_k, t_k))_{l\in\mathbb{N}}$, $(D^2\phi^k_{u,l}(x_k, t_k))_{l\in\mathbb{N}}$ and

$(D^2\phi^k_{v,l}(y_k,s_k))_{l\in\mathbb{N}}$ is converging and the limits (which depend on k) are converging again, there exists a constant $M>0$ such that for all $k\geq k_0$ and all $l\in\mathbb{N}$ we have

$$\left|\nabla\phi^k_{u,l}(x_k,t_k)\right|+\left|D^2\phi^k_{u,l}(x_k,t_k)\right|\leq M;\ \left|\nabla\phi^k_{v,l}(y_k,s_k)\right|+\left|D^2\phi^k_{v,l}(y_k,s_k)\right|\leq M.$$

Therefore, by assumption there is some constant $K>0$, independent of k and l such that

$$\partial_t\phi^k_{u,l}(x_k,t_k)\leq K \quad \text{and} \quad \partial_t\phi^k_{v,l}(y_k,s_k)\geq -K \quad \forall l\in\mathbb{N}.$$

Hence, letting first $l\to+\infty$ in particular

$$|\beta_k(t_k-s_k)|\leq K.$$

Consequently – up to possibly passing to another subsequence – we can assume that there exists some $a\in\mathbb{R}$ such that

$$\beta_k(t_k-s_k)\to a \quad \text{as } k\to\infty.$$

The desired sequences are now obtained by a diagonal argument, so that we will have $\phi_{u,k}=\phi^k_{u,l_k}$ and $\phi_{v,k}=\phi^k_{v,l_k}$ for a suitable subsequence $(l_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$. Finally, let us discuss the case, when $(\hat{x},\hat{y},\hat{t})$ is not necessarily a strict global maximum of Φ_α. In this case we note that the function

$$\widetilde{\Phi}_\alpha(x,y,t):=u(x,t)-v(y,t)-\frac{\alpha}{2}|x-y|^2+(|x-\hat{x}|^4+|y-\hat{y}|^4+\left|t-\hat{t}\right|^4),$$

has a strict global maximum at $(\hat{x},\hat{y},\hat{t})$. Up to minor modifications, we can replace Φ_α by $\widetilde{\Phi}_\alpha$ in the previous arguments and deduce the desired conclusion also in this case. □

4.4. Uniqueness of Flat Flows.

The comparison principle now immediately yields a characterization of our flat flows.

Proposition 4.4.1. *Let u be a flat flow as in Definition 3.3.1 and let v be a viscosity supersolution of equation (4.7) in $\Omega\times[0,+\infty[$ such that $u=v$ on $\partial_p(\Omega\times[0,+\infty[)$. Then $u\leq v$ in $\Omega\times[0,+\infty[$.*

Proof. By Proposition 4.2.2 we know that u is a viscosity solution of (4.7). Hence the proposition follows from applying the comparison principle (Proposition 4.3.1). □

In other words, u (as in the above proposition) is the smallest viscosity supersolution of (4.7) or, yet in other words, u is the smallest viscosity supersolution of

$$u_t-\sqrt{1+|\nabla u|^2}\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}=0,$$

which satisfies $u\geq\psi$.

As a consequence we also deduce uniqueness of flat flows.

Corollary 4.4.2. *Let* $(u^{h_k})_{k\in\mathbb{N}}$ *and* $(u^{h_l})_{l\in\mathbb{N}}$ *be two sequences of approximating discrete flows obtained from Theorem 3.1.3 such that* $u^{h_k} \to u_1$ *as* $(k \to \infty)$ *and* $u^{h_l} \to u_2$ *as* $(l \to \infty)$ *in* $L^\infty(\Omega \times [0,T])$ *for any* $T > 0$*. Then* $u_1 = u_2$*. Moreover, we deduce that* $(u^h)_{h>0}$ *converges along the full sequence as* $h \to 0$*.*

Proof. We apply the previous proposition once with $u = u_1$, $v = u_2$ to deduce $u_1 \leq u_2$ and then vice versa to get $u_2 \leq u_1$. The second statement follows form the just established uniqueness and the observation that it is possible to construct a converging subsequence out of every given sequence $(u^{h_\nu})_{\nu\in\mathbb{N}}$ of approximate flows. □

Remark 4.4.1. In the language of minimizing movements, we thus showed, that the flat flow u is also minimizing movement – in contrast with generalized minimizing movements. For example see [39] for more details on this notion.

A. Quasilinear Equations

In this first part of the appendix we collect some classical results on the regularity of solutions to second-order, uniformly elliptic, quasilinear equations. As in the linear case, the basic estimates are derived using the *difference quotient method* so we recall here also the relevant statements about difference quotients. For a proof we refer to Section 7.11 in [68].

Definition A.1. Let $\Omega', \Omega \subset \mathbb{R}^n$ both be open and such that $\Omega' \subset\subset \Omega$. Then, for $u : \Omega \to \mathbb{R}$, $h \in \mathbb{R}$ with $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$ and $1 \leq i \leq n$ we define the *i-th difference quotient of size h of u* as:

$$D_i^h u(x) := \frac{u(x + he_i) - u(x)}{h} \quad x \in \Omega'. \tag{A.1}$$

Proposition A.1. *Let $\Omega' \subset\subset \Omega \subset \mathbb{R}^n$ both be open.*

i) Suppose $1 \leq p < +\infty$, $u \in W^{1,p}(\Omega)$. Then we have:

$$\|D_i^h u\|_{L^p(\Omega')} \leq \|\partial_i u\|_{L^p(\Omega)} \quad \forall |h| \leq \operatorname{dist}(\Omega', \partial\Omega). \tag{A.2}$$

ii) Suppose $1 < p < +\infty$, $u \in L^p(\Omega')$. If for any $1 \leq i \leq n$ there are constants $C_i \geq 0$ such that:

$$\left\|D_i^h u\right\|_{L^p(\Omega')} \leq C_i \quad \forall |h| \leq \operatorname{dist}(\Omega', \partial\Omega), \tag{A.3}$$

then $u \in W^{1,p}(\Omega')$ and $\|\partial_i u\|_{L^p(\Omega')} \leq C_i$ for every i.

iii) Suppose $u, v \in L^2(\Omega)$, $\operatorname{supp}(u) \subset \Omega' \subset\subset \Omega$ and $|h| \leq \operatorname{dist}(\Omega', \partial\Omega)$. Then we have the following integration by parts formula:

$$\int_\Omega (D_i^h u) v \,\mathrm{d}x = -\int_\Omega u(D_i^{-h} v) \,\mathrm{d}x. \tag{A.4}$$

We start with the interior L^2-estimate.

Theorem A.2. *Let $\Omega \subset \mathbb{R}^n$ open and bounded. Suppose that $a \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ is such that Da is elliptic, i.e. for every $R > 0$ there exists $\Lambda_1 = \Lambda_1(R) > 0$*

$$\Lambda_1 |\xi|^2 \leq \langle Da(p)\xi, \xi\rangle \quad \forall p \in B_R(0)\, \forall \xi \in \mathbb{R}^n, \tag{A.5}$$

and uniformly bounded, i.e. there exists $\Lambda_2 \geq 0$

$$|Da| := \sqrt{\sum_{1 \leq i,j \leq n} \left|\frac{\partial a_i}{\partial p_j}(p)\right|^2} \leq \Lambda_2 \quad \forall p \in \mathbb{R}^n. \tag{A.6}$$

Furthermore, let $f \in L^2(\Omega)$ *and* $u \in W^{1,\infty}(\Omega)$ *be a weak solution of*

$$-\mathrm{div}(a(\nabla u)) = f. \tag{A.7}$$

Then $u \in W^{2,2}_{loc}(\Omega)$ *and for every* $\Omega' \subset\subset \Omega$ *we have the estimate*

$$\|u\|_{W^{2,2}(\Omega')} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}), \tag{A.8}$$

where C *depends only on* n, Λ_1, Λ_2 *and* $\mathrm{dist}(\Omega', \partial\Omega)$ *where* $\Lambda_1 = \Lambda_1(\|\nabla u\|_{L^\infty(\Omega)})$.

Proof. Recall that being a weak solution of (A.7) means that:

$$\int_\Omega \langle a(\nabla u), \nabla v\rangle = \int_\Omega f v \quad \forall v \in W^{1,2}_0(\Omega). \tag{A.9}$$

Fix an arbitrary $\Omega' \subset\subset \Omega$ and choose some open Ω'' with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Let $\eta \in C^\infty_c(\Omega'')$ be a cut-off function with the properties:

$$\eta_{|\Omega'} = 1, \quad 0 \leq \eta \leq 1 \,\text{and}\, |\nabla \eta| \leq 2/d,$$

where $d := \mathrm{dist}(\Omega', \partial\Omega'')$. Additionally we impose that $2d \geq \mathrm{dist}(\Omega', \partial\Omega)$. It is easy to check that for h small enough and $1 \leq j \leq n$ the function $v := -D^{-h}_j(\eta^2 D^h_j u)$ has compact support in Ω and belongs to $W^{1,2}_0(\Omega)$. Inserting v in (A.9) and using integration by parts for D^{-h}_j we get:

$$\int_\Omega \langle D^h_j(a(\nabla u)), \nabla(\eta^2 D^h_j(u))\rangle = \int_\Omega f v. \tag{A.10}$$

Observe that:

$$\begin{aligned}
D^h_j(a(\nabla u))(x) &= \frac{1}{h}(a(\nabla u(x+he_j)) - a(\nabla u(x))) \\
&= \frac{1}{h}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}[a(t\nabla u(x+he_j) + (1-t)\nabla u(x)]\mathrm{d}t \\
&= \int_0^1 [Da(t\nabla u(x+he_j) + (1-t)\nabla u(x)](D^h_j \nabla u(x))\mathrm{d}t \\
&= A(x)\, D^h_j \nabla u(x).
\end{aligned}$$

Where $A(x) := A(x;h) := \int_0^1 Da(t\nabla u(x+he_j) + (1-t)\nabla u(x))\mathrm{d}t$. Inserting this in the previous equation we end up with

$$\underbrace{\int_\Omega \langle A\, D^h_j \nabla u, \nabla(\eta^2 D^h_j(u))\rangle}_{=B_1} = \underbrace{\int_\Omega f v}_{B_2}. \tag{A.11}$$

Before further estimating both sides of this equation let us note that the matrix A is uniformly elliptic and inherits the boundedness from Da. Indeed, for any $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$ we get by using the linearity of integration

$$\begin{aligned}\langle A(x)\,\xi, \xi\rangle &= \int_0^1 \langle Da(t\nabla u(x+he_j) + (1-t)\nabla u(x))\xi, \xi\rangle \,\mathrm{d}t \\ &\geq \int_0^1 \Lambda_1 |\xi|^2 \,\mathrm{d}t = \Lambda_1 |\xi|^2,\end{aligned}$$

where $\Lambda_1 = \Lambda_1(\|\nabla u\|_{L^\infty(\Omega)})$. The boundedness of the coefficients follows analogous. Indeed, for every $x \in \Omega$ we have

$$\sum_{1\leq i,j\leq n} |A_{ij}(x)|^2 \leq \int_0^1 \sum_{1\leq i,j\leq n} \left|\frac{\partial a^i}{\partial p_j}(t\nabla u(x+he_j) + (1-t)\nabla u(x))\right| \mathrm{d}t \leq \Lambda^2.$$

We start estimating the left hand side of (A.11). By the chain rule we immediately get

$$B_1 = \int_\Omega \langle AD_j^h \nabla u, 2\eta(\nabla\eta) D_j^h u\rangle + \int_\Omega \langle AD_j^h \nabla u, \eta^2 D_j^h \nabla u\rangle.$$

The second integral can easily be estimated from below by using ellipticity:

$$\int_\Omega \langle AD_j^h \nabla u, \eta^2 D_j^h \nabla u\rangle \geq \Lambda_1 \int_\Omega \eta^2 \left|D_j^h \nabla u\right|^2,$$

while the first one can be bounded as follows:

$$\begin{aligned}\left|\int_\Omega \langle AD_j^h \nabla u, 2\eta(\nabla\eta) D_j^h u\rangle\right| &\leq 4\int_{\Omega''} \left|\eta A D_j^h \nabla u\right| \frac{1}{d}\left|D_j^h u\right| \\ &\leq 4\varepsilon\,\Lambda_2^2 \int_{\Omega''} \eta^2 \left|D_j^h \nabla u\right|^2 + \frac{1}{\varepsilon d^2}\int_{\Omega''}\left|D_j^h u\right|^2.\end{aligned}$$

Choosing $\varepsilon = \frac{\Lambda_1}{8\Lambda_2^2}$ and applying Proposition A.1 *i*) we get

$$\left|\int_\Omega \langle AD_j^h \nabla u, 2\eta(\nabla\eta) D_j^h u\rangle\right| \leq \frac{\Lambda_1}{2}\int_\Omega \eta^2 \left|D_j^h \nabla u\right| + \frac{8\Lambda_2^2}{\Lambda_1 d^2}\int_\Omega |\partial_j u|^2.$$

Altogether we have

$$B_1 \geq \frac{\theta}{2}\int_\Omega \eta^2 \left|D_j^h \nabla u\right|^2 - \frac{C\Lambda_2^2}{d^2\Lambda_1}\int_\Omega |\partial_j u|^2. \tag{A.12}$$

Now let us analyze B_2. As an auxiliary estimate we compute:

$$\begin{aligned}\int_\Omega |v|^2 &= \int_{\Omega''} \left|D_j^{-h}(\eta^2 D_j^h u)\right|^2 \\ &\leq \int_{\Omega''} \left|\partial_j(\eta^2 D_j^h u)\right|^2, \quad \text{by } i) \text{ in Proposition A.1} \\ &= \int_{\Omega''} \left|2\eta(\partial_j\eta) D_j^h u + \eta^2 D_j^h(\partial_j u)\right|^2 \\ &\leq C\int_{\Omega''} |\partial_j \eta|^2 \left|D_j^h u\right|^2 + \eta^2 \left|D_j^h \partial_j u\right|^2, \quad \text{as } |\eta|^4 \leq |\eta|^2, \\ &\leq \frac{C}{d^2}\int_\Omega |\partial_j u|^2 + C\int_\Omega \eta^2 \left|D_j^h \partial_j u\right|^2, \quad \text{again by } i) \text{ in Proposition A.1.}\end{aligned}$$

Hence, applying Young's inequality for any $\varepsilon > 0$ we get

$$|B_2| \leq \int_\Omega |f||v| \leq \frac{1}{4\varepsilon} \int_\Omega |f|^2 + \varepsilon \left(\frac{C}{d^2} \int_\Omega |\partial_j u|^2 + C \int_\Omega \eta^2 \left| D_j^h \partial_j u \right|^2 \right).$$

Choosing $\varepsilon = \frac{\Lambda_1}{4C}$ one has:

$$|B_2| \leq \frac{C}{\Lambda_1} \int_\Omega |f|^2 + \frac{\Lambda_1}{4d^2} \int_\Omega |\partial_j u|^2 + \frac{\Lambda_1}{4} \int_\Omega \eta^2 \left| D_j^h \partial_j u \right|^2. \tag{A.13}$$

Combining (A.11), (A.12) and (A.13) we finally get

$$\frac{\Lambda_1}{2} \int_\Omega \eta^2 \left| D_j^h \nabla u \right|^2 - \frac{C\Lambda_2^2}{d^2 \Lambda_1} \int_\Omega |\partial_j u|^2 \leq \frac{C}{\Lambda_1} \int_\Omega |f|^2 + \frac{\Lambda_1}{4d^2} \int_\Omega |\partial_j u|^2 + \frac{\Lambda_1}{4} \int_\Omega \eta^2 \left| D_j^h \partial_j u \right|^2,$$

from which we conclude

$$\int_{\Omega'} \left| D_j^h \nabla u \right|^2 \leq \int_\Omega \eta^2 \left| D_j^h \nabla u \right|^2 \leq \frac{C}{\Lambda_1^2} \int_\Omega |f|^2 + (1 + \frac{\Lambda_2^2}{\Lambda_1^2}) \frac{C}{d^2} \int_\Omega |\partial_j u|^2.$$

□

Having established the interior $W^{2,2}$-regularity we can now differentiate equation (A.7) which then allows us to apply De Giorgi-Nash-Mosers's regularity result and derive the $C^{1,\alpha}$-Hölder-regularity of u. We will restrict ourselves to the mean curvature operator.

Theorem A.3. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $a : \mathbb{R}^n \to \mathbb{R}^n$ be defined as*

$$a(p) = \frac{p}{\sqrt{1 + |p|^2}}. \tag{A.14}$$

Furthermore, let $f \in L^p(\Omega)$ for $p > n$ and $u \in W^{1,\infty} \cap W^{2,2}_{loc}(\Omega)$ a weak solution of (A.7)*. Then there exists $0 < \alpha < 1$ such that $u \in C^{1,\alpha}_{loc}(\Omega)$ and for every $\Omega' \subset\subset \Omega$ we have the estimate*

$$\sup_{x,y \in \Omega'} \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\alpha} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}), \tag{A.15}$$

where the constants C and α depend only on $n, p, \|\nabla u\|_{L^\infty}$ and $\operatorname{dist}(\Omega', \partial\Omega)$.

Proof. For fixed $1 \leq k \leq n$ let us test equation (A.7) with $\partial_{x_k}\phi$ where $\phi \in C_c^\infty(\Omega)$ to get

$$\int_\Omega a(\nabla u) \cdot \nabla(\partial_{x_k}\phi) \, \mathrm{d}x = \int_\Omega f \partial_{x_k}\phi \, \mathrm{d}x.$$

Since u is twice weakly differentiable and the since the vector field a is smooth, also $a(\nabla u)$ is once (weakly) differentiable. Therefore, by partial integration, on the left hand side we can pass the differentiation in x_k onto $a(\nabla u)$ and obtain (using the chain rule)

$$-\int_\Omega Da(\nabla u) \nabla(\partial_{x_k} u) \cdot \nabla\phi \, \mathrm{d}x = \int_\Omega f \partial_{x_k}\phi \, \mathrm{d}x.$$

Since ϕ was arbitrary this last equation is saying that $w := \partial_{x_k} u \in W^{1,2}_{loc}(\Omega)$ is a weak solution of the *linear* equation

$$-\mathrm{div}(A\nabla w) = \partial_{x_k} f,$$

where $A(x) := Da(\nabla u(x))$. The theorem is now a consequence of the regularity theorem of De Giorgi-Nash-Moser, see for instance Theorem 8.24 in [68], which gives the Hölder-regularity of w. By definition this gives us the desired continuity for u, as k was arbitrary. For the sake of completeness, let us check that A satisfies the hypotheses of the regularity theorem. First of all, we note that symmetry of A follows from symmetry of Da which can be written as the second derivative of the smooth function

$$p \mapsto \sqrt{1+|p|^2}.$$

Boundedness is also immediate as $|Da(p)| \leq 2$ for every $p \in \mathbb{R}^n$. Finally to get the uniform ellipticity from the following observation. The eigenvalues of $Da(p)$ are given by (see Lemma B.1)

$$\left(1+|p|^2\right)^{-\frac{1}{2}} \quad \text{and} \quad \left(1+|p|^2\right)^{-\frac{3}{2}}.$$

Therefore, using the Lipschitz bound on u we deduce the uniform ellipticity

$$\langle A(x)\xi, \xi\rangle \geq \left(1+\|\nabla u\|^2_{L^\infty(\Omega)}\right)^{-\frac{3}{2}} |\xi|^2.$$

□

A combination of the previous two results and the classical L^p- and Schauder estimates yield now an L^p- and a Schauder version of Theorem A.2.

Theorem A.4. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and define a as in* (A.14)*. Furthermore, let $f \in L^p(\Omega)$ for some $p > n$ and $u \in W^{1,\infty}(\Omega)$ be a weak solution of*

$$-\mathrm{div}(a(\nabla u)) = f. \tag{A.16}$$

Then $u \in W^{2,p}_{loc}(\Omega)$ and for every $\Omega' \subset\subset \Omega$ we have the estimate

$$\|u\|_{W^{2,p}(\Omega')} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Omega)}), \tag{A.17}$$

where C depends only on n, p, $\|\nabla u\|_{L^\infty(\Omega)}$ and $\mathrm{dist}(\Omega', \partial\Omega)$.

Proof. By the previous two theorems we deduce that $u \in C^{1,\alpha}_{loc}(\Omega)$ which allows us then to apply the classical Calderon-Zygmund estimates, see for instance section 9.5 in [68]. □

Theorem A.5. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and define a as in* (A.14)*. Furthermore, let $H \in W^{1,\infty}(\Omega \times \mathbb{R})$ and $u \in W^{1,\infty} \cap C^{1,\alpha}_{loc}(\Omega)$ is a weak solution of*

$$\mathrm{div}(a(\nabla u)) = H(\cdot, u). \tag{A.18}$$

Then, for every $0 < \beta < 1$, $u \in C^{2,\beta}_{loc}(\Omega)$.

Proof. As before, we note that after taking one derivative, we obtain that $w := \partial_{x_k} u$ is a weak solution of

$$-\text{div}(A\nabla w) = \partial_{x_k} f,$$

where $A(x) := Da(\nabla u(x))$ and $f(x) = H(x, u(x))$. Since both A and f are locally $C^{0,\alpha}$ we can apply the Schauder-estimates for equations in divergence form (see for instance Theorem 5.19 in [65]) to deduce that Dw is in $C^{0,\alpha}_{loc}(\Omega)$ and consequently that u is in $C^{2,\alpha}_{loc}(\Omega)$. But now we observe that due to the embedding $C^{2,\alpha}_{loc}(\Omega) \hookrightarrow C^{1,\beta}_{loc}(\Omega)$, for any $0 < \beta < 1$ we get that A is locally $C^{0,\alpha}$ and f is locally $C^{1,\beta}$. Applying once more the Schauder-estimates concludes the proof. □

Remark A.1. Assuming that $\partial\Omega$ is smooth, Theorem A.2, Theorem A.3, Theorem A.4 and Theorem A.5 have global counterparts, i.e. the respective regularity holds up to the boundary as long as the boundary data is sufficiently regular.

We will conclude this first part of the appendix by citing two versions of the (weak) comparison principle.

Proposition A.6. *Let $v, w \in W^{1,\infty}(\Omega)$ and suppose that they weakly satisfy*

$$\text{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) \le \text{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right).$$

Then it holds

$$\inf_{\Omega}(v-w) = \inf_{\partial\Omega}(v-w).$$

In particular, if $v \ge w$ on $\partial\Omega$ then $v \ge w$ in Ω.

Proof. As v and w are Lipschitz continuous, we trivially have

$$\inf_{\Omega}(v-w) = \inf_{\overline{\Omega}}(v-w) \le \inf_{\partial\Omega}(v-w).$$

For the opposite inequality, we start by setting

$$a(p) := \frac{p}{\sqrt{1+|p|^2}} \quad \text{for } p \in \mathbb{R}^n.$$

By the fundamental theorem of calculus it is easy to see that $u := v - w$ weakly solves

$$\text{div}\,(A(x)\nabla u) \le 0, \tag{A.19}$$

with

$$A(x) := \int_0^1 Da(t\nabla v(x) + (1-t)\nabla w(x))\mathrm{d}t.$$

It is then straightforward to check that A is bounded and uniformly elliptic. Hence, by the weak maximum principle (see for instance [68, Theorem 8.1]) we get

$$\inf_{\Omega} u \ge \inf_{\partial\Omega} u.$$

We will give a quick proof of this variant of the maximum principle here. Testing the inequality (A.19) with $v = \max\{m-u, 0\}$ where $m := \inf_{\partial\Omega} u$, we get

$$\int_{\{u<m\}} \langle A(x)\nabla u, \nabla u\rangle \, \mathrm{d}x \leq 0.$$

On the other hand, the uniform ellipticity of A immediately implies that for some $\lambda > 0$

$$0 \leq \lambda \int_{\{u<m\}} |\nabla u|^2 \, \mathrm{d}x \leq \int_{\{u<m\}} \langle A(x)\nabla u, \nabla u\rangle \, \mathrm{d}x,$$

so that altogether we get

$$\int_{\{u<m\}} |\nabla u|^2 \, \mathrm{d}x = 0.$$

Noting that on the boundary of $\{u < m\}$ we have $u = m$, this implies that $u = m$ in all of $\{u < m\}$ which can only be true if $|\{u < m\}| = 0$. In other words, if $\inf_\Omega u \geq \inf_{\partial\Omega} u$. $\square$

Remark A.2. To be precisely, Theorem 8.1 in [68] states that $\inf_\Omega u \geq \inf_{\partial\Omega} u^-$, with $u^- = \min\{u, 0\}$. However, by using the (vertical) translation-invariance of our equation we can always assume that $u \leq 0$ and hence $u^- = u$.

Proposition A.7. *Suppose $\Omega \subset \mathbb{R}^n$ is open and bounded, $a : \Omega \to \mathbb{R}^{n\times n}_{sym}$, bounded, measurable and uniformly elliptic, this means $\exists \lambda > 0$ such that*

$$\langle \xi, a(x)\xi\rangle > \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \quad \forall x \in \Omega.$$

Let $F \in L^p(\Omega; \mathbb{R}^n)$ for some $p > n$. Suppose $u \in W^{1,2}(\Omega)$ is a sub-solution of

$$\operatorname{div}(a(x)\nabla u) = \operatorname{div}(F),$$

which means

$$\int_\Omega \langle \nabla v, a\nabla u\rangle \, \mathrm{d}x \leq \int_\Omega \langle F, \nabla v\rangle \, \mathrm{d}x \quad v \in C^1_c(\Omega).$$

Then we have

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + \frac{C}{\lambda}\|F\|_{L^p(\Omega)},$$

where $C = C(n, p, \Omega) > 0$ and $u^+ = \max\{0, u\}$. Analogous, if $u \in W^{1,2}(\Omega)$ is a super-solution of $\operatorname{div}(a(x)\nabla u) = \operatorname{div}(F)$, then we have

$$\inf_\Omega u \geq \inf_{\partial\Omega} u^- - \frac{C}{\lambda}\|F\|_{L^p(\Omega)},$$

with the same C and where and $u^- = \min\{0, u\}$.

Proof. Theorem 8.16 in [68]. $\square$

B. Miscellaneous Results

Mean Curvature Operator

Lemma B.1. *Let $a(p) := (1+|p|^2)^{-\frac{1}{2}}p$ for $p \in \mathbb{R}^n$. Then $Da(p)$ is a symmetric matrix with eigenvalues $(1+|p|^2)^{-\frac{3}{2}}$ and $(1+|p|^2)^{-\frac{1}{2}}$. Moreover, for every $R > 0$, a satisfies*

$$(a(p) - a(q)) \cdot (p - q) \geq (1 + |R|^2)^{-\frac{3}{2}} |p - q|^2 \quad \forall p, q \in B_R(\Omega).$$

Proof. We have

$$Da(p) = \frac{1}{(1+|p|^2)^{\frac{2}{3}}} \left((1+|p|^2)\mathrm{id} - p \otimes p \right).$$

Noting that $(p \otimes p)p = |p|^2 p$ and $(p \otimes p)v = 0$ for any v perpendicular to p, it is straightforward to check that the eigenspaces of Da are given by $V_1 = \mathbb{R}p$ (dimension 1, with eigenvalue $(1+|p|^2)^{-\frac{3}{2}}$) and $V_2 = V_1^{\perp}$ (dimension $n-1$, with eigenvalue $(1+|p|^2)^{-\frac{1}{2}}$). Hence, for any $p \in B_R(0)$ we know that

$$(Da(p)\xi) \cdot \xi \leq (1 + |R|^2)^{-\frac{3}{2}} |\xi|^2 \quad \forall \xi \in \mathbb{R}^n.$$

Using this lower bound it follows that for all $p, q \in B_R(0)$ we have

$$(a(p) - a(q)) \cdot (p - q) = \int_0^1 Da(\underbrace{tp + (1-t)q}_{\in B_R(0)})(p - q)\,\mathrm{d}t \cdot (p - q) \geq (1 + |R|^2)^{-\frac{3}{2}} |p - q|^2.$$

□

Equicontinuity

Let (X, d) and (Y, ρ) be a two metric spaces. Recall that a family $\mathcal{F}$ of functions from X to Y is called *uniformly equicontinuous* if

$$\forall \varepsilon > 0\ \exists \delta > 0 : d(x_1, x_2) \leq \delta \implies \rho(f(x_1), f(x_2)) \leq \varepsilon \quad \forall x_1, x_2 \in X\ \forall f \in \mathcal{F}.$$

Proposition B.2. *Let (X, d) be a compact metric space and $f_n, f : X \to Y$ such that*

i) $\{f_n : n \in \mathbb{N}\}$ is uniformly equicontinuous,

ii) $f_n(x) \to f(x)$ point-wise as $(n \to +\infty)$ $\forall x \in X$.

Then $f_n \to f$ uniformly $(n \to +\infty)$.

Proof. Let $\varepsilon > 0$ and choose $\delta > 0$ such that whenever for every $x_1, x_2 \in X$ we have:

$$d(x_1, x_2) \leq \delta \implies \rho(f_n(x_1), f_n(x_2)) \leq \varepsilon \quad \forall n \in \mathbb{N}.$$

Observe that by using $i)$ we can pass into the limit to derive that additionally:

$$d(x_1, x_2) \leq \delta \implies \rho(f(x_1), f(x_2)) \leq \varepsilon.$$

Finally, by compactness of X we choose $x_1, \dots x_N$ such that $X = \cup_{i=1}^N B_\delta(x_i)$ and we let n_0 be large enough so that

$$\rho(f_n(x_i), f(x_i)) \leq \varepsilon \quad \forall i = 1, \dots, N \ \forall n \geq n_0.$$

Fix now any $x \in X$ and let $1 \leq i_0 \leq N$ be such that $x \in B_\delta(x_{i_0})$. Combining the last three estimates and using the triangle inequality we get:

$$\rho(f_n(x), f(x)) \leq \rho(f_n(x), f_n(x_{i_0})) + \rho(f_n(x_{i_0}), f(x_{i_0})) + \rho(f(x_{i_0}), f(x)) \leq 3\varepsilon.$$

□

Remark B.1. One can easily weaken the hypothesis $ii)$ to point-wise convergence on a dense subset of X and still deduce the uniform convergence.

Sets of Finite Perimeter

In this section we collect some results about sets of finite perimeter.

Lemma B.3. *Let E be a set of finite perimeter in $\mathbb{R}^n$ then there exists a Borel set F such that $|E \Delta F| = 0$ and*

$$\{x \in \mathbb{R}^n : 0 < |E \cap B_r(x)| < \omega_n r^n \ \forall r > 0\} = \partial F.$$

We call such a set a point-wise representative *of E.*

Proof. See [97, Proposition 12.19]. □

Remark B.2. Note that we cannot speak about *the* point-wise representative of a set of finite perimeter. Consider for instance $E = B_1(0)$ then both E and $\overline{E}$ are point-wise representatives of E. However for our purpose this way of modifying a set of finite perimeter is sufficient as we show in the next lemma.

Lemma B.4. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, suppose F_1 and F_2 are point-wise representatives of E. Then we have*

$$\mathrm{dist}_{F_1}(x) = \mathrm{dist}_{F_2}(x) \quad \forall x \in \mathbb{R}^n,$$

and consequently

$$\mathrm{sdist}_{F_1}(x) = \mathrm{sdist}_{F_2}(x) \quad \forall x \in \mathbb{R}^n.$$

Proof. Observe that by definition, we have $\partial F_1 = \partial F_2$. Now we recall that in $\mathbb{R}^n$ the distance of a set $F \subset \mathbb{R}^n$ is attained at a point of its boundary. Combining these two observation yields the desired result. A more direct way of arguing we be to verify that $F_1 \Delta F_2 \subset \partial F_1$. □

Next we recall the Isoperimetric Inequality in $\mathbb{R}^n$.

Theorem B.5 (Isoperimetric Inequality)**.** *If E is a Lebesgue measurable set in $\mathbb{R}^n$ with $|E| < +\infty$ then*

$$n\omega_n^{1/n}|E|^{(n-1)/n} \leq \mathrm{Per}(E).$$

Equality holds if and only if $|E \triangle B_r(x)| = 0$ for some $x \in \mathbb{R}^n$ and $r > 0$.

Proof. See [97, Theorem 14.1]. □

As a direct consequence we get an equivalent and sometimes more useful formulation.

Corollary B.6. *If E is a Lebesgue measurable set in $\mathbb{R}^n$ with $|E| = |B_r(0)|$ for some $r > 0$ then*

$$\mathrm{Per}(B_r(0)) \leq \mathrm{Per}(E).$$

Equality holds if and only if $|E \Delta B_r(x)| = 0$ for some $x \in \mathbb{R}^n$ and $r > 0$.

Convex Functions

Suppose $\Omega \subset \mathbb{R}^n$ is open and convex. Recall that we call a real-valued function $u : \Omega \to \mathbb{R}$ convex if

$$u((1-\lambda)x + \lambda y) \leq (1-\lambda)u(x) + \lambda u(y) \quad \forall x, y \in \Omega \, \forall 0 < \lambda < 1.$$

Moreover, u is called strictly convex if the strict inequality holds whenever $x \neq y$. First of all, we recall that if u is C^1 then monotonicity of ∇u implies convexity.

Lemma B.7. *Let $\Omega \subset \mathbb{R}$ open and convex, $u \in C^1(\Omega)$ with*

$$\langle \nabla u(y) - \nabla u(x), y - x \rangle \geq 0 \quad \forall x, y \in \Omega \quad \textit{(monotonicity)}.$$

Then u is convex. Moreover, if we even have

$$\langle \nabla u(y) - \nabla u(x), y - x \rangle > 0 \quad \forall x \neq y \in \Omega \quad \textit{(strict monotonicity)},$$

then u is strictly convex.

Proof. Let $x, y \in \Omega$. As u is C^1 we get by the fundamental theorem of calculus that

$$u(y) = u(x) + \int_0^1 \langle \nabla u(x + t(y-x)), y - x \rangle \, \mathrm{d}t.$$

By the monotonicity of ∇u we know that

$$\langle \nabla u(x+t(y-x)) - \nabla u(x)), x+t(y-x)-x\rangle \geq 0,$$

which by $t \geq 0$ implies $\langle \nabla u(x+t(y-x), y-x\rangle \geq \langle \nabla u(x), y-x\rangle$. Thus we get

$$u(y) \geq u(x) + \langle \nabla u(x), y-x\rangle. \tag{B.1}$$

Note that x and y where chosen arbitrarily. It is straightforward to derive now the convexity of u from this inequality. Indeed for any $0 < \lambda < 1$ we set $z := (1-\lambda)x + \lambda y$. Then applying inequality (B.1) once with x and z and once with y and z we get

$$\begin{aligned} u(z) + \lambda\langle \nabla u(z), (x-y)\rangle &\leq u(x),\\ u(z) + (1-\lambda)\langle \nabla u(z), (y-x)\rangle &\leq u(y). \end{aligned}$$

Multiplying the first inequality with $(1-\lambda)$ and the second one with λ and adding both of them we derive the desired inequality. Finally, it is easy to see that in the case of strict monotonicity, if $x \neq y$, then instead of (B.1) we also get a strict inequality from which we can then derive strict convexity by the same argument that we used to derive convexity before. □

As a consequence of the previous lemma we get a convexity criteria for $C^{1,1}$ functions. Recall that by Rademacher's theorem a Lipschitz function is differentiable at almost every point of its domain. Thus, for $u \in C^{1,1}$ the second derivative D^2u is defined almost everywhere.

Lemma B.8. *Let $\Omega \subset \mathbb{R}^n$ open and convex, $u \in C^{1,1}(\Omega)$. Then u is convex if and only if $D^2u \geq 0$ almost everywhere.*

Proof. Suppose that $u \in C^{1,1}(\Omega)$ is convex and fix some open $V \subset\subset \Omega$. Now one can construct an approximating sequence $(u_k)_{k\in\mathbb{N}}$ in $C^\infty(V)$ where $u_k = u * \rho_k$ for a sequence of mollifiers $(\rho_k)_{k\in\mathbb{N}}$. Since convexity is stable under mollification we get $D^2u_k \geq 0$ almost everywhere in V for each $k \in \mathbb{N}$. In particular, for almost every $x \in V$

$$D^2u_k(x) \geq 0 \quad \forall k \in \mathbb{N}.$$

As Du is Lipschitz, the point-wise (a.e.) defined $D^2u \in L^\infty(\Omega)$ is also the weak second derivative of u and we have $D^2u_k = D^2u * \rho_k \to D^2u$ point-wise almost everywhere in V. Consequently $D^2u \geq 0$ at almost every point in V. The conclusion follows now immediately since we can exhaust Ω by the countable sequence of open sets, for instance, $V_l = \{x \in \Omega : \mathrm{dist}_{\partial\Omega}(x) > \frac{1}{l}\}$.

Suppose now that $D^2u \geq 0$ a.e. and let $x, y \in \Omega$ and choose $\varepsilon > 0$. As $\nabla u \in \mathrm{Lip}(\Omega)$ we can find a $\delta > 0$ such that $|\nabla u(x) - \nabla u(z)| < \varepsilon$ for every $z \in B_\delta(x)$. Without loss of generality we may assume that $\delta < \varepsilon$. Now we pick $z \in B_\delta(x)$ such that $D^2u(z+t(y-z))$ exists and is non-negative for almost every $0 < t < 1$. Note that it is always possible to find such a z since otherwise we would contradict our assumption on D^2u on a set of positive measure. Hence, we get

$$\nabla u(y) = \nabla u(z) + \int_0^1 D^2u(z + t(y - z)) \cdot (y - z)\,\mathrm{d}t.$$

And consequently

$$\langle \nabla u(y) - \nabla u(z), y - z\rangle = \int_0^1 \langle y - z, D^2u(z + t(y - z)) \cdot (y - z)\rangle \geq 0,$$

where we used the non-negativity of D^2u along the segment. Finally, we see that

$$\begin{aligned}
&\langle \nabla u(y) - \nabla u(x), y - x\rangle \\
&= \langle \nabla u(y) - \nabla u(z), y - x\rangle + \langle \nabla u(z) - \nabla u(x), y - x\rangle \\
&\geq \langle \nabla u(y) - \nabla u(z), y - z\rangle + \langle \nabla u(y) - \nabla u(z), z - x\rangle - |y - x|\varepsilon \\
&\geq 0 - (2\mathrm{Lip}(\nabla u) + |y - x|)\varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we can conclude by applying the first part of Lemma B.7. □

Remark B.3. Note that in general it is not true that if $u \in C^{1,1}$ is strictly convex, then one has that $D^2u > 0$ almost everywhere. To see this, consider for instance a closed, nowhere dense set $K \subset]0, 1[$ of positive measure (for instance a so called *fat* Cantor set). Then, for $x \in]0, 1[$ let $u(x) := \int_0^x \int_0^y \mathrm{dist}_K(z)\,\mathrm{d}z\,\mathrm{d}y$. Note that for every $x \in K$ we get $u''(x) = \mathrm{dist}_K(x) = 0$. However, since K is nowhere dense u' is strictly increasing and thus u is strictly convex.

Semijets

In this short section we provide some elementary facts about semijets, i.e. sub- and superjets as introduced in Section 4. To begin with, we recall their definitions.

Definition B.1. For $V \subset \mathbb{R}^N$ open, $u \in C(V)$ and $x_0 \in V$ we define the set of *second order superjets (of* u *at* x_0*)* as

$$\begin{aligned}
J^{2,+}u(x_0) := \{(p, X) \in \mathbb{R}^N \times \mathbb{R}^{N\times N}_{sym} :\ &\text{as } x \to x_0 \text{ we have} \\
&u(x) \leq u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + o(|x - x_0|^2)\ \}.
\end{aligned}$$

The usage of the symbol $o(|x - x_0|^2)$ has to be understood in the following way: There exists a function $h : V \to \mathbb{R}$, continuous at x_0 with $h(x_0) = 0$ such that for all $x \in V$:

$$u(x) \leq u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + h(x)|x - x_0|^2.$$

Analogously, we define the set of *second order subjets (of u at x_0)* as

$$J^{2,-}u(x_0) := \{(p, X) \in \mathbb{R}^N \times \mathbb{R}^{N\times N}_{sym} : \text{ as } x \to x_0 \text{ we have} \\ u(x) \geq u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + o(|x - x_0|^2) \}.$$

It is straightforward to check that equivalently we could have used the identity

$$J^{2,-}u(x_0) = -J^{2,+}(-u)(x_0), \tag{B.2}$$

to define $J^{2,-}u(x_0)$.

The notion of sub- and superjets can be seen as a generalization of derivatives of u up two second order. In fact, semijets can be used two characterize second order differentiability of u.

Lemma B.9. *The function u is twice differentiable at x_0 if and only if there exists $(p, X) \in \mathbb{R}^N \times \mathbb{R}^{N\times N}_{sym}$ such that*

$$J^{2,+}u(x_0) \cap J^{2,-}u(x_0) = \{(p, X)\}.$$

Moreover, if u is twice differentiable at x_0 we have $(p, X) = (\nabla u(x_0), D^2u(x_0))$.

Proof. First of all, we claim that the intersection of $J^{2,+}u(x_0)$ and $J^{2,-}u(x_0)$ is either empty or a singleton (this is essentially saying that derivatives are unique). In order to prove this claim, due to the anti-symmetry of the order relation in $\mathbb{R}^{N\times N}_{sym}$ it suffices to check the following simpler claim: If $(p, X) \in J^{2,-}u(x_0)$ and $(q, Y) \in J^{2,+}u(x_0)$ then we get $p = q$ and $X \leq Y$. Fix therefore any such (p, X) and (q, Y) and recall that by definition we have (for $x \to x_0$)

$$u(x) \leq u(x_0) + \langle q, x - x_0\rangle + \frac{1}{2}\langle Y(x - x_0), x - x_0\rangle + o(|x - x_0|^2),$$

and

$$u(x) \geq u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + o(|x - x_0|^2).$$

Substituting $x = y + x_0$ we hence deduce the estimate (as $y \to 0$)

$$\langle p, y\rangle + \frac{1}{2}\langle Xy, y\rangle \leq \langle q, y\rangle + \frac{1}{2}\langle Yy, y\rangle + o(|y|^2). \tag{B.3}$$

Setting $y = \varepsilon v$, where $\varepsilon > 0$ and $v \in \mathbb{R}^N$ and then dividing by ε we get (for $\varepsilon \to 0$)

$$\langle p, v\rangle + \frac{\varepsilon}{2}\langle Xv, v\rangle \leq \langle q, v\rangle + \frac{\varepsilon}{2}\langle Yv, v\rangle + \frac{o(\varepsilon^2)}{\varepsilon}.$$

Letting $\varepsilon \to 0$ we deduce that for every $v \in \mathbb{R}^N$ we have $\langle p, v\rangle \leq \langle q, v\rangle$ which implies $p = q$. Therefore, we can further deduce from (B.3) that

$$\langle Xy, y\rangle \leq \langle Yy, y\rangle + o(|y|^2).$$

As before, we will again set $y = \varepsilon v$ and this time divide by ε^2 to get that (as $\varepsilon \to 0$):

$$\langle Xv, v\rangle \le \langle Yv, v\rangle + \frac{o(\varepsilon^2)}{\varepsilon^2}.$$

Again the claim follows by letting $\varepsilon \to 0$. Assume now that u is twice differentiable at x_0. It suffices to verify that $(\nabla u(x_0), D^2u(x_0)) \in J^{2,+}u(x_0) \cap J^{2,-}u(x_0)$ which follows immediately from Taylor's theorem and the differentiability of u at x_0. For the reverse implication we assume that $\{(p, X)\} = J^{2,+}u(x_0) \cap J^{2,-}u(x_0)$. Hence there exist two functions $h_1, h_2 : V \to \mathbb{R}$, continuous at x_0 and $h_1(x_0) = h_2(x_0) = 0$ such that for all $x \in V$

$$\begin{aligned} u(x) &\le u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + h_1(x)|x - x_0|^2, \\ u(x) &\ge u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + h_2(x)|x - x_0|^2. \end{aligned}$$

Setting for $x \in V$

$$h(x) := u(x) - u(x_0) - \langle p, x - x_0\rangle - \frac{1}{2}\langle X(x - x_0), x - x_0\rangle,$$

we hence deduce that for all $x \in V$

$$h_1(x) \le h(x) \le h_2(x).$$

Thus also h has to be continuous and vanishing at 0 which implies that u is twice differentiable at x_0 with $\nabla u(x_0) = p$ and $D^2u(x_0) = X$. □

Even if u is only continuous, by using an appropriate class of *test functions* it is still possible to characterize the sets of sub- and superjets of u via the (classical) derivatives of C^2-functions.

Lemma B.10. *If $u \in C(V)$ and $x_0 \in V$ we have the following characterization of semijets via C^2-tests:*

$$J^{2,+}u(x_0) = \{(\nabla\phi(x_0), D^2\phi(x_0)) \mid \phi \in C^2(V),\ \phi \ge u,\ \phi(x_0) = u(x_0)\},$$

and

$$J^{2,-}u(x_0) = \{(\nabla\phi(x_0), D^2\phi(x_0)) \mid \phi \in C^2(V),\ \phi \le u,\ \phi(x_0) = u(x_0)\}.$$

Proof. In both cases, the fact that every C^2-test gives rise to either a super- or a subjet follows from Taylor's theorem. The other inclusion requires some work and we will essentially follow the ideas used in the proof of Lemma 3.2.7 in [67]. Without loss of generality, let us assume that $x_0 = 0$ and let $(p, X) \in J^{2,+}u(0)$. By definition we know that there exists a function $h : V \to \mathbb{R}$, continuous at 0 with $h(0) = 0$ such that for all $x \in V$:

$$u(x) \le u(0) + \langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle + h(x)|x|^2.$$

If $h \in C^2(V)$ we would be done but all we know so far is the continuity of h at the origin. We will now regularize h in three steps. First of all, for $r \geq 0$ we define h_1 by

$$h_1(r) := \sup\{|h(x)| \ : \ |x| \leq r\}.$$

By definition h_1 is monotone and satisfies $h(x) \leq h_1(|x|)$. In the next step we construct a continuous function by defining $h_2 : [0, +\infty[\to [0, +\infty[$ via first setting $h_2(0) := 0$ and for any $k \in \mathbb{Z}$

$$h_2(2^k) := h_1(2^{-k+1}),$$

and then extending it by linear interpolation for arbitrary $r \in]0, +\infty[$. Observe that also for h_2 we have $h(x) \leq h_2(|x|)$ for every $x \in V$ and h_2 is also monotone. Finally, we define $h_3 : [0, +\infty[\to [0, +\infty[$ via

$$h_3(r) := \int_r^{2r} \int_s^{2s} h_2(t) \, \mathrm{d}t \, \mathrm{d}s.$$

By monotonicity of h_2 we immediately get

$$h_3(r) \geq \int_r^{2r} s h_2(s) \, \mathrm{d}s \geq r^2 h_2(r),$$

and thus $h(x)|x|^2 \leq h_3(|x|)$ for any $x \in V$. Moreover, $h_3 \in C^2([0, +\infty[)$ with $h_3(0) = h_3'(0) = h_3''(0) = 0$ and hence (see the auxiliary lemma right after this proof) the function

$$V \ni x \mapsto h_3(|x|) \in [0, +\infty[,$$

is also of class $C^2(V)$ and vanishes together with its first and second derivative at the origin. We can therefore close our argument by setting

$$\phi(x) := u(0) + \langle p, x \rangle + \frac{1}{2}\langle Xx, x \rangle + h_3(|x|),$$

since by construction $\phi \geq u$ in V, $\phi(0) = u(0)$, $\nabla\phi(0) = p$ and $D^2\phi(0) = X$. By the same method, we can construct C^2-tests for any element in $J^{2,-}u(x_0)$. $\square$

Lemma B.11. *Let $h \in C^2([0, +\infty[)$ with $h(0) = h'(0) = h''(0) = 0$. Then the map $\phi : \mathbb{R}^N \to \mathbb{R}$, $x \mapsto h(|x|)$ belongs to $C^2(\mathbb{R}^N)$ and both $\nabla\phi$ and $D^2\phi$ vanish at the origin.*

Proof. By the chain rule and the smoothness of $x \mapsto |x|$ away from the origin we know that $\phi \in C^2(\mathbb{R}^N \setminus \{0\})$ and for $x \in \mathbb{R}^N \setminus \{0\}$ we have

$$\nabla\phi(x) = h'(|x|)\frac{x}{|x|} \quad \text{and } D^2\phi(x) = h''(|x|)\frac{x \otimes x}{|x|^2} + h'(|x|)\frac{|x|^2 I_N - x \otimes x}{|x|^3}.$$

Observe that the singularity in the second term of $D^2\phi$ can be controlled as follows: for any $1 \leq j, k \leq N$ we have

$$h'(|x|)\frac{|x|^2\delta_{jk} - x_j x_k}{|x|^3} \leq \frac{|h'(|x|)|}{|x|} \to h''(0) = 0 \quad \text{as } x \to 0.$$

Therefore we get

$$\nabla\phi(x) \to 0 \quad \text{and } D^2\phi(x) \to 0 \quad \text{as } x \to 0.$$

Thus, we only have to show that ϕ is twice differentiable at the origin and that these derivatives are equal to 0. As far as the first derivative is concerned, for any $1 \leq k \leq N$ and $\varepsilon > 0$ we get

$$\frac{\phi(\varepsilon e_k) - \phi(0)}{\varepsilon} = \frac{h(\varepsilon)}{\varepsilon} \to h'(0) = 0 \quad \text{as } x \to 0.$$

Hence $\phi \in C^1(\mathbb{R}^N)$. To show that $\phi \in C^2(\mathbb{R}^N)$ let $1 \leq j, k \leq N$ and compute

$$\frac{(\partial_k\phi)(\varepsilon e_j) - (\partial_k\phi)(0)}{\varepsilon} = \frac{h'(\varepsilon)\delta_{jk}}{\varepsilon} \to h''(0) = 0 \quad \text{as } x \to 0.$$

□

C. A Penalization Approach

In this final part of the appendix we would like to roughly outline how the penalization method can be used to obtain higher regularity. Rather than giving a complete and self-contained derivation of these results we would like to give the reader an idea of the method and a flavor of the computations that have to be made. We closely follow the work of Rupflin-Schnürer [116] and adapt it to our setting where necessary.

Notation and Preliminary Results

The basic idea is to consider a family of flows which are not necessarily respecting the obstacle. However, an additional forcing term will penalize the penetration of the obstacle and as we let the penalization become stronger we expect that the penalized flow converges to one that respects the obstacle. The flow which is allowed to penetrate the obstacle will be as smooth as the forcing term allows and it will be chosen in a way to obtain uniform (in the penalization parameter) $C^{1,1}$-in-space estimates which then pass over to the limiting flow.

As in the main part of the thesis, we let $\Omega \subset \mathbb{R}^n$ be a bounded, open and convex set with smooth boundary. For simplicity, henceforth we will assume that we have zero boundary data. The obstacle $\psi : \Omega \to \mathbb{R}$ is supposed to be at least $C^{1,1}$-regular and satisfies $\psi < 0$ on $\partial\Omega$. The initial surface is the graph of some $u_0 \in C^2(\overline{\Omega})$ and satisfies $u_{0|\partial\Omega} = 0$ and $u_0 \geq \psi$ in Ω. Moreover, let us also assume that $u_0 \geq 0$.

Let us now consider $F_\varepsilon : \Omega \times [0, +\infty[\to \mathbb{R}^{n+1}$ such that $F_\varepsilon(\cdot, t)$ is an immersion for all $t \geq 0$ with

$$\{F_\varepsilon(x,0) \,:\, x \in \Omega\} = \{(x, u_0(x)) \,:\, x \in \Omega\}, \tag{C.1}$$

$$F_\varepsilon(x,t) = (x,0) \qquad \forall x \in \partial\Omega\ \forall t \geq 0, \tag{C.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} F_\varepsilon(x,t) = \Delta_{M_t} F(x,t) + \alpha_\varepsilon(F_\varepsilon(x,t))\nu_{M_t}(x,t) \qquad \text{in } \Omega\times]0,+\infty[. \tag{C.3}$$

Here $\alpha_\varepsilon(p) = \beta_\varepsilon(\mathrm{sdist}_\psi(p))$, $\beta_\varepsilon(s) = \beta(\frac{s}{\varepsilon})$, for some smooth β with $\beta = 0$ on $[0, +\infty[$, $\beta'' > 0$ and $\beta' < 0$ on $]-\infty, 0]$. Finally, M_t denotes the image of $F(\cdot, t)$ and Δ_{M_t} and ν_{M_t} denote the Laplacian and normal (upward) associated to M_t respectively.

Definition C.1. For F_ε as above we define the associated *graphical representation* as $u_\varepsilon(x,t) = \Pi_{n+1}(F_\varepsilon(\phi^t_\varepsilon(x), t))$, where $\Pi_{n+1} : \mathbb{R}^{n+1} \to \mathbb{R}$ is the projection on the last component and $\phi^t_\varepsilon(x)$ is the inverse of the diffeomorphism $\varphi^t_\varepsilon : \Omega \ni x \mapsto \Pi(F_\varepsilon(x,t))$, with $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denoting the projection onto the first n components.

Next, we need to settle some notation. Henceforth, $F = F_\varepsilon$ denotes an embedding map as in (C.1) to (C.3) but ε shall be suppressed in the notation.

Definition C.2. We define the metric as

$$g_{ij}(x,t) = \partial_i F(x,t) \cdot \partial_j F(x,t), \quad 1 \le i,j \le n,$$

and since $F(\cdot,t)$ is an embedding we can define (g^{ij}) as the inverse of (g_{ij}). Note, that here and in the following ∂_i is referring to the usual partial derivative in the x_i-direction in $\Omega \subset \mathbb{R}^n$. The second fundamental form is the tensor

$$A_{ij} = -\partial_i \partial_j F \cdot \nu_{M_t}, \quad 1 \le i,j \le n,$$

and the mean curvature vector associated to M_t can be expressed as

$$\vec{H}_{M_t} = (\Delta_{M_t} F)\nu_{M_t}. \tag{C.4}$$

Finally, we set

$$v := \frac{1}{\nu_{M_t} \cdot e_{n+1}}, \quad v_\psi := \frac{1}{\nu_\psi \cdot e_{n+1}}, \tag{C.5}$$

where ν_ψ is the normal vector to the graph of ψ, seen as a submanifold in $\mathbb{R}^{n+1}$ and the height of F is defined as

$$U := F_\varepsilon \cdot e_{n+1}. \tag{C.6}$$

Of course, the graphical representation satisfies the usual mean curvature flow equation for graphs.

Lemma C.1. *Let u_ε be the graphical representation associated to F_ε. Then $u_\varepsilon = 0$ on $\partial\Omega$, $u_\varepsilon(\cdot,0) = u_0$, and*

$$\partial_t u_\varepsilon = \sqrt{1+|\nabla u_\varepsilon|^2}\left(\operatorname{div}\left(\frac{\nabla u_\varepsilon}{\sqrt{1+|\nabla u_\varepsilon|^2}}\right) + \alpha_\varepsilon(\cdot,u_\varepsilon)\right). \tag{C.7}$$

Proof. To simplify the notation, let us drop all ε-indices in this proof. Multiplying (C.3) with $\nu_{M_t}(x,t)$ and using (C.4) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}F(x,t)\cdot\nu_{M_t}(x,t) = \vec{H}_{M_t}\cdot\nu_{M_t}(x,t) + \alpha(F(x,t)).$$

Since $F_\varepsilon(x,t) = (\varphi_\varepsilon^t(x), u_\varepsilon(\varphi_\varepsilon^t(x),t))$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F(x,t) &= \left(\frac{\partial}{\partial t}\varphi^t(x), \nabla u(\varphi^t(x),t)\cdot\frac{\partial}{\partial t}\varphi^t(x) + \frac{\partial u}{\partial t}(\varphi^t(x),t)\right)\\ &= \left(\frac{\partial}{\partial t}\varphi^t(x), \nabla u(\varphi^t(x),t)\cdot\frac{\partial}{\partial t}\varphi^t(x)\right) + \left(0, \frac{\partial u}{\partial t}(\varphi^t(x),t)\right).\end{aligned}$$

Considering the map $\Psi : \Omega \ni y \mapsto (y, u(y,t)) \in M_t$ and consequently, for every $y \in \Omega$ we get that $D\Psi_y : \mathbb{R}^n \to T_{\Psi(y)}M_t$. In particular, $D\Psi_y(\frac{\partial}{\partial t}\varphi^t(x)) \in T_{\Psi(y)}M_t$. Thus, choosing $y = \varphi^t(x)$ and noting that $\Psi(y) = F(x,t)$ and $\nu_{M_t}(x,t) \perp T_{F(x,t)}M_t$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}F(x,t)\cdot\nu_{M_t}(x,t) = \frac{\frac{\partial}{\partial t}u}{\sqrt{1+|\nabla u|^2}}(\varphi^t(x),t).$$

The lemma is now a consequence of the following well known relation for the scalar mean curvature

$$\vec{H}_{M_t} \cdot \nu_{M_t}(x,t) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)(\varphi^t(x),t).$$

□

As in [116] one can derive the following evolution equations of the various geometric quantities. In the following, we use ∇_{M_t} to denote the tangential gradient associated to M_t and $\nabla_i\nabla_j$ refers to the covariant derivative, see [41, Appendix A] for details.

Lemma C.2. *For $F = F_\varepsilon$ as in (C.1) to (C.3) and U as in* (C.6) *we have*

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \Delta_{M_t} F + \alpha_\varepsilon(F)\nu_{M_t}, \tag{C.8}$$

$$\frac{\mathrm{d}g_{ij}}{\mathrm{d}t} = -2(H-\alpha_\varepsilon)A_{ij}, \tag{C.9}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right)\nu = |A|^2\nu - \nabla_{M_t}\alpha_\varepsilon, \tag{C.10}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right)v = -|A|^2 v - \frac{2}{v}|\nabla_{M_t} v|^2 + v^2\langle \nabla_{M_t}\alpha_\varepsilon, e_{n+1}\rangle, \tag{C.11}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right)|A|^2 = -2|\nabla_{M_t}A|^2 + 2|A|^4 - 2\alpha_\varepsilon A_i^k A_j^i A_k^j - 2\nabla_i\nabla_j\alpha_\varepsilon A^{ij}, \tag{C.12}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right)\alpha_\varepsilon = \beta_\varepsilon'\alpha_\varepsilon\langle\nu_\psi,\nu_{M_t}\rangle - \beta_\varepsilon''|\Pi_{TM}\nu_\psi|^2 - \beta_\varepsilon'(\Delta_\psi)_{ij}g^{ij}, \tag{C.13}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right)U = \frac{\alpha_\varepsilon}{v}. \tag{C.14}$$

We remind the reader that we several times used the Einstein convention for summation over repeated indices as well as the shorthand $A_j^i := g^{ik}A_{kj}$ concerning the raising of indices.

Remark C.1. At every point of $M_t \cap \{(x,z) : z \le \psi(x)\}$ sufficiently close to the obstacle with $v \ge v_\psi$ we have that

$$\langle \nabla_{M_t}\alpha_\varepsilon, e_{n+1}\rangle \le 0,$$

where v_ψ is the gradient function to the level sets

Proof. By the chain rule we get $\nabla_{M_t}\alpha_\varepsilon = \beta_\varepsilon'\nabla_{M_t}\mathrm{sdist}_\psi$ and for points sufficiently close to the obstacle (more precisely in a tubular neighborhood) we know that

$$\nabla_{M_t}\mathrm{sdist}_\psi P_{TM}(\nu_\psi) = (\nu_\psi - \langle\nu_\psi,\nu_{M_t}\rangle\nu_{M_t}).$$

Consequently, since $\beta_\varepsilon' \le 0$ we get

$$\langle\nabla_{M_t}\alpha_\varepsilon, e_{n+1}\rangle = \beta_\varepsilon'(\frac{1}{\nu_\psi} - \frac{\langle\nu_\psi,\nu\rangle}{v}) \le 0.$$

□

Derivation of the Estimates

First of all, we can easily derive a bound on the depth of penetration.

Lemma C.3. *Let F_ε as in (C.1) to (C.3) and let u_ε be the associated graphical representation. Then there exists a constant C_1, independent of ε such that*

$$u_\varepsilon(x,t) \geq \psi(x) - C_1\varepsilon \quad \forall x \in \Omega\, \forall t \geq 0.$$

Proof. Again we use the convention $u = u_\varepsilon$. It suffices to check that for sufficiently large $C_1 > 0$, $\psi - C_1\varepsilon$ is a subsolution of the equation

$$u_t = \sqrt{1+|\nabla u|^2}\left(\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + \alpha_\varepsilon(x, u_\varepsilon(x))\right).$$

Then, the claim follows by Lemma C.1 and the standard comparison principle for parabolic equations [93, Theorem 9.1]. Hence, we need to show the existence of $C_1 > 0$ such that for suitable $C(\psi) > 0$

$$0 \leq \underbrace{\sqrt{1+|\nabla \psi|^2}\operatorname{div}\left(\frac{\nabla \psi}{\sqrt{1+|\nabla \psi|^2}}\right)}_{\geq -C_1(\psi)} + \sqrt{1+|\nabla\psi|^2}\alpha_\varepsilon(x, \psi(x) - C_1\varepsilon).$$

Since $\sqrt{1+|\nabla\psi|^2} \geq 1$, it suffices to choose $C_1 > 0$ large enough such that

$$C(\psi) \leq \beta\left(\frac{\operatorname{sdist}_\psi(x, \psi(x) - C_1\varepsilon)}{\varepsilon}\right).$$

By Lemma 3.4.5, for some constant $\hat{C}$, depending only on the Lipschitz constant of ψ, we get $\operatorname{sdist}_\psi(x, \psi(x) - C_1\varepsilon) \geq -\hat{C}C_1\varepsilon$. Recalling the monotonicity of β we therefore have to choose

$$C_1 \geq \frac{-\beta^{-1}(C(\psi))}{\hat{C}}.$$

□

As a corollary of this estimate we can prove the following consistency.

Lemma C.4. *Let F and ϕ^t be the uniform limits of F_ε and ϕ^t_ε respectively. We denote by $u, u_\varepsilon : \Omega \times [0, +\infty[\to \mathbb{R}$ the associated graphical representation. Then $u_\varepsilon \to u$ uniformly and u is a viscosity solution of* (4.7)*.*

Remark C.2. In particular, using the results from Chapter 4, u coincides with the flat flow that we obtained from the discretization scheme.

Proof of Lemma C.4. The uniform convergence of u_ε is a direct consequence of the fact that $u_\varepsilon(x,t) = \Pi_{n+1}(F_\varepsilon(\phi_\varepsilon^t(x),t))$ and the convergence of F_ε, ϕ_ε^t. Combining now the uniform convergence of u_ε with Lemma C.3 we derive that $u \geq \psi$. Hence, we can continue as in the proof of Proposition 4.2.2 and use (as in [116]) an analog of [26, Proposition 2.9] on the closedness of families of viscosity solutions. □

Next we prove an a priori gradient bound.

Proposition C.5 (C^1 estimate)**.** *Recalling the definitions of v and U in* (C.5) *and* (C.6)*, we get that*

$$\sup_{\Omega\times[0,+\infty[} U^2 v \leq C_2,$$

for some suitable $C_2 > 0$, not depending on ε.

Proof. Let us compute the evolution of $w := U^2 v$. We get

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) U^2 v = v\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) U^2 + U^2\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) v - 2\nabla_{M_t} U^2 \nabla_{M_t} v.$$

Using the evolution of U we obtain

$$v\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) U^2 = 2U\alpha_\varepsilon - 2v|\nabla_{M_t} U|^2.$$

Likewise, we compute

$$U^2\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) v = U^2(-|A|^2 v - \frac{2}{v}|\nabla_{M_t} v|^2 + v^2\langle\nabla_{M_t}\alpha_\varepsilon, e_{n+1}\rangle),$$

and noting that at points where w is maximal, $2U\nabla_{M_t} U v = -U^2\nabla_{M_t} v$, we have

$$-2\nabla_{M_t} U^2 \nabla_{M_t} v = -4U\nabla_{M_t} U \cdot v = 2U^2|\nabla_{M_t} v|^2 \frac{1}{v}.$$

Altogether, we therefore get

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) U^2 v \leq 2U\alpha_\varepsilon - 2v\left|\nabla_{M_t} U^2\right| + v^2\langle\nabla_{M_t}\alpha_\varepsilon, e_{n+1}\rangle.$$

Due to Remark C.1, for $v \geq C = C(\psi)$, the last term is negative and can hence be dropped as well. Furthermore, a direct computation gives us

$$|\nabla_{M_t} U|^2 = 1 - \frac{1}{v^2},$$

and thus, for $v \geq 2$ we can further estimate

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) U^2 v \leq 2U\alpha_\varepsilon - v.$$

Finally, recalling Lemma C.3, we also know that

$$\|\alpha_\varepsilon\|_{L^\infty} \le C_1,$$

so that $2U\alpha_\varepsilon \le 2C_1 \sup_\Omega u_0$.

Therefore, we showed that for some $C_2 > 0$ we get that $v \ge C_2$ implies

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) U^2 v < 0.$$

On the other hand, suppose that $w \ge M := C_2 \sup u_0$ (in which case we would have $v \ge C_2$). Let then $t_0 = \inf\{t > 0 : \left\|U^2(\cdot,t)v(\cdot,t)\right\|_{L^\infty} > M\}$. Then we can find $x_0 \in \Omega$ such that

$$w(x_0,t_0) \ge M \quad \text{and} \quad \forall x \in \Omega : \ w(x_0,t_0) \ge w(x,t_0).$$

In particular, this implies

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) U^2 v \ge 0,$$

which is a contradiction. □

We now come to the derivation of the estimate on the second fundamental form. As can be seen upon expanding (C.9), the evolution equation for $|A|^2$ contains terms involving the penalization term α_ε together with its first and second derivatives. Moreover, we can no longer expect those terms to have a sign. Thus, similar to the proof of the curvature bound in [41], Rupflin-Schnürer consider a modified second fundamental form quantity to which they then apply the maximum principle. For the detailed computations we again refer to [116] and just highlight the core quantity of these computations here. Without proof we state the following result which is an analog of [116, Lemma 8.2].

Lemma C.6. *There exist numbers $\gamma, k, R > 0$ such for*

$$G := h(v^2)e^{\gamma\alpha_\varepsilon}|A|^2, \quad \text{where } h(y) := ye^{ky},$$

we have

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) G \le &-\frac{1}{h}\langle\nabla_{M_t}h, \nabla_{M_t}G\rangle - \frac{k}{8}(he^{\gamma\alpha_\varepsilon}|\nabla_{M_t}A|^2 + G|A|^2 + G|\nabla_{M_t}v|^2)\\ &-\left(\frac{\gamma}{8}\beta''_\varepsilon|P_{TM}\nu_\psi|^2 + \frac{1}{2}|\beta'_\varepsilon|\langle\nu,\nu_\psi\rangle_+\right)G,\end{aligned}$$

at every point $p \in M_t$ with $|A(p)| \ge R$.

Using this result, by reasoning similarly to the proof of Lemma C.2.3 one gets

Proposition C.7 (Bound on $|A|^2$)**.** *There exists $C_3 > 0$, not depending on ε, with*

$$\sup_{\Omega\times[0,+\infty[} U^4|A|^2 \le C_3.$$

Taking now the previous proposition for granted, one can then derive the following theorem.

Theorem C.8. *Let u be the solution of* (4.7). *Then, for every $T > 0$ we have $u \in C^{1,1;0,1}(\Omega \times [0,T])$, i.e. $C^{1,1}$ in space and $C^{0,1}$ in time.*

Sketch of the proof. Letting $\varepsilon \to 0$, (up to establishing the uniform convergence of F_ε and ϕ_ε^t) we already know from Lemma C.4 that u_ε converges towards the viscosity solution of (4.7). Using Propositions C.5 and C.7 which are uniform in ε we derive the desired (interior) spacial regularity on u by compactness. Near the boundary $\partial\Omega$ we note that by our assumptions on u_0 and ψ we have $u(x,t) \geq 0 > \psi(x)$ for all $t > 0$ and whenever $x \in \Omega$ is close enough to $\partial\Omega$. Therefore, $\alpha_\varepsilon(x, u_\varepsilon(x,t)) = 0$ for such points (x,t) which means that u_ε is solving (graphical) mean curvature flow. Hence, arguing as in the proof of [77, Theorem 2.1] we can derive a bound on $\sup_{\partial\Omega\times[0,T]} |\nabla u_\varepsilon|$. By the parabolic maximum principle and the evolution equation for v (C.11), we then obtain uniform bounds on $|\nabla u|$. This makes sure that the equation becomes uniformly parabolic and we can apply the standard theory to deduce smoothness of the solution near the boundary. The time-regularity follows now immediately by using equation (C.7).

We emphasize once more, that many steps were skipped in this sketch. In particular, we skip the approximation of the signed distance function (which in general is just Lipschitz) by smooth functions via mollification which in the present case has to be combined with a cut-off argument near the boundary. Moreover, we did not touch on the subtle existence questions for the approximate flows but just took it for granted as our main interest was to highlight the derivation of the $C^{1,1}$-estimates. Finally, we also note that we discarded problems that would occur with nonhomogeneous boundary conditions. □

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